On the Gauge Invariance of Wigner-Dunkl Quantum Mechanics in the Presence of a Constant Magnetic Field

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Abstract

In this paper we consider a charged Wigner-Dunkl quantum system in the presence of a constant magnetic field. It is shown that this system obeys gauge invariance if minimally coupled to a vector potential following the Dunkl-Maxwell relations. A family of vector potentials, which generate the constant magnetic field, is constructed explicitly. The gauge invariance of the Wigner-Dunkl quantum system is established with a gauge transformation exhibiting a deformed unitarity. For vector potentials following the standard Maxwell relations it is not possible to establish gauge invariance for the Wigner-Dunkl quantum system.

Keywords: Deformed Heisenberg Algebra, Wigner-Dunkl Quantum Mechanics, Gauge Invariance

1 Introduction

In 1950 Wigner [1] put forward a classic question: Can one derive the corresponding commutation relations for physical quantities from the classic equations of motion? Wigner used the harmonic oscillator as an example to study this problem within the matrix mechanics and concluded that the popular commutation relation $[x, p] = \mathrm{i}\hbar$ may not always be the most general one. After the publication of Wigner's work Yang [2], with the advice and support of Born, studied Wigner's article within the wave mechanics and found that the commutation relation always holds as long as the wave function conditions were properly used, including a more rigorous definition of Hilbert space and a more rigorous series expansion. In his work, Yang introduced a reflection operator R with the property $Rf(x) = f(-x)$, and concluded that a more general momentum operator of the form $p = -\mathrm{i}\hbar(\partial_x - (c/2x)R)$ was admissible with c being an arbitrary constant. Then, in 1989 the Dunkl operator [3] was proposed, which is a combination of differential and difference operators involving the reflection operator R, and turned out to be closely related to the coordinate representation of Wigner's original findings as shown by Yang [2].

During the last decades the deformed Heisenberg algebra [4] has been reconsidered in the context of quantum mechanics, the so-called Wigner-Dunkl quantum mechanics. The deformed Heisenberg algebra with reflection, introduced inexplicitly by Wigner and explicitly by Yang, found very important developments. This algebra was generalized in the form of trilinear commutation relations for the case of various degrees of freedom and led to the appearance of the notion of parastatistics, see refs. [5, 6, 7]. Later, hidden nonlinear supersymmetry

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was revealed in purely parabosonic oscillator systems in ref. [8]. Parastatistics played an important role in the discovery of "color" degrees of freedom in hadron physics and in the formulation of $su(3)$ gauge theory of QCD [9]. The harmonic oscillator in connection with the deformed Heisenberg algebra in presence of a reflection operator was studied in refs. [10, 11, 12]. The Coulomb problem in two and three dimensions was discussed in [13] and [14]. Supersymmetric aspects of Wigner-Dunkl quantum systems were studied, for example, in [15, 16, 17, 18, 19].

Thermodynamic properties of Dunkl-bosonic systems in the context of Dunkl-statistical mechanics are discussed in [20, 21, 22, 23]. In ref. [20] the thermodynamics of boson systems related to Dunkl differential–difference operators is presented. The ideal Bose gas and blackbody radiation in the framework of Dunkl formalism have been studied in ref. [21], and ref. [22] deals with the condensation of ideal Bose gas in a gravitational field in the framework of Dunkl-statistic. The thermal properties of relativistic-Dunkl oscillators are investigated in ref. [23], and then the same authors have studied the three-dimensional Dunkl-Klein-Gordon equations under Coulomb potential in [24].

Other recent applications of the deformed Heisenberg algebra are related to the construction of the bosonized form of the Dirac equation in various dimensions [25]. Here, in addition, the case of the electromagnetic coupling of a general form using the reflection operators was considered and generalization to the non-Abelian case was discussed. In ref. [26] the deformed Heisenberg algebra with reflection was related to non-commutative quantum mechanics. The deformed Heisenberg algebra with reflection in presence of the deformation parameter was used for the construction of non-unitary anyons which interpolate between bosons and fermions in ref. [27]. In ref. [28], the two-dimensional deformed Heisenberg algebra with reflection was used for the generation of the $(2+1)D$ massive anyons by compactification on a circle of the formal $(3+1)D$ massless fields with fractional helicity.

A study on Dunkl graphene in a constant magnetic field can be found in ref. [29]. Dunkl-Maxwell equations are discussed in [30], where in essence the partial differential operators are replaced by Dunkl derivatives. Whereas ref. [30] includes a discussion on the Dunkl variant of gauge transformations it remains at a purely classical level.

It is the objective of the current work to investigate the gauge invariance of a charged Wigner-Dunkl quantum system minimally coupled to a constant magnetic field utilizing the Dunkl-Maxwell formulation of [30]. This paper is organised as follows. In section 2 we will briefly review the Wigner-Dunkl quantum formalism and minimally couple that to an external magnetic field utilising Dunkl-Maxwell relations. Then, in section 3, we will limit ourselves to a constant magnetic field. Here a family of vector potentials is constructed of which all members represent a constant magnetic field. The associated gauge function is found, transforming two members of the family into each other. In section 4, we will establish the gauge invariance of the charged Wigner-Dunkl quantum system for the constant magnetic field configuration. As the gauge transformation is nonlocal a detailed discussion on the transformation of wave functions is required. This is done in section 5. Section 6 then considers the same setup, however, within the standard non-deformed Maxwell approach for vector potentials. It is shown that no gauge invariance for the Wigner-Dunkl quantum system can be achieved within the usual Maxwell formalism. That is, the Dunkl-Maxwell relations [30] are vital for gauge invariance of Wigner-Dunkl quantum mechanics. A summary of our findings is presented in section 7.

2 Wigner-Dunkl Hamiltonian with magnetic field

Wigner-Dunkl quantum mechanics in three dimensions is based on a deformed Heisenberg algebra containing reflection operators

$$[X_i, P_j] = \mathrm{i}\hbar(1 + 2\nu_i R_i)\delta_{ij}\,, \qquad \nu_i > -\frac{1}{2}\,, \qquad i,j = 1,2,3\,. \tag{1}$$

In the above X_i and P_i denote the components of position and momentum operator, respectively. The reflection operators R_j are the parity operators acting on the plane perpendicular to the x_j-axis. That is, they have the following properties when acting on functions f defined on $\mathbb{R}^3$.

$$\begin{aligned} R_1 f(x_1,x_2,x_3) &= f(-x_1,x_2,x_3)\,, \\ R_2 f(x_1,x_2,x_3) &= f(x_1,-x_2,x_3)\,, \\ R_3 f(x_1,x_2,x_3) &= f(x_1,x_2,-x_3)\,. \end{aligned} \tag{2}$$

The real deformation parameters ν_i are called Dunkl parameters and are bounded from below as indicated in (1). In the coordinate representation the position operator X_i is, as usual, represented by the coordinate x_i. However, the components of the momentum operator $P_j := -\mathrm{i}\hbar D_j$ are represented by Dunkl derivatives

$$D_j := \frac{\partial}{\partial x_j} + \nu_j\,(1+R_j)\,\frac{1}{x_j} = \frac{\partial}{\partial x_j} + \frac{\nu_j}{x_j}\,(1-R_j)\;, \qquad j=1,2,3\,. \tag{3}$$

In the following we will exclusively work in the coordinate representation and make use of the vector notation $\boldsymbol{x} := (x_1,x_2,x_3)^T$ and $\boldsymbol{P} := (P_1,P_2,P_3)^T$ when appropriate.

The free Wigner-Dunkl Hamiltonian for a particle of mass $m>0$ is given by

$$H := \frac{\boldsymbol{P}^2}{2m} \tag{4}$$

and acts on the weighted Hilbert space $\mathcal{H} := L^2(\mathbb{R}^3, \mathrm{d}\mu)$ with scalar product

$$(\varphi,\psi) := \int_{\mathbb{R}^3} \mathrm{d}\mu(\boldsymbol{x})\,\varphi^*(\boldsymbol{x})\psi(\boldsymbol{x})\,, \qquad \varphi,\psi\in\mathcal{H}\,. \tag{5}$$

and weighted measure

$$\mathrm{d}\mu(\boldsymbol{x}) := \prod_{j=1}^{3} \mathrm{d}x_i |x_i|^{2\nu_i}\,. \tag{6}$$

Let us note that the components P_j of the momentum operator are Hermitian on $\mathcal{H}$. See the appendix, where we explicitly show that

$$(\varphi, P_j^\dagger\psi) := (P_j\varphi,\psi) = (\varphi, P_j\psi) \qquad \text{for all} \qquad \varphi,\psi\in\mathcal{H}\,. \tag{7}$$

The objective of this paper is to investigate the gauge invariance of a charged Wigner-Dunkl particle with charge e, $e<0$ for the electron, in the presence of an external magnetic field. In doing so let us briefly recall the Dunkl variant of Maxwell's equations following ref. [30], where electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ are derivable from a scalar potential ϕ and a vector potential $\boldsymbol{A}$ via the below relations with $\boldsymbol{D} := (D_1,D_2,D_3)^T$

$$\boldsymbol{E} = -\boldsymbol{D}\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\,, \qquad \boldsymbol{B} = \boldsymbol{D}\times\boldsymbol{A}\,. \tag{8}$$

That is, the standard Maxwell relations are modified such that the partial derivatives with respect to the coordinates are replaced by the corresponding Dunkl derivatives. Focusing on a purely external magnetic field we will from now on consider a vanishing scalar potential $\phi=0$ and a time-independent vector potential $\partial\boldsymbol{A}/\partial t = 0$. As in standard quantum mechanics we propose a minimal coupling scheme. Hence, the Wigner-Dunkl Hamiltonian of a charge particle with charge e and mass $m>0$ is given by

$$H := \frac{1}{2m}\left(\boldsymbol{P} - \frac{e}{c}\boldsymbol{A}\right)^2\,, \tag{9}$$

where c denotes the speed of light.

3 Vector potential and gauge transformation

In this section we will construct the vector potential associated with a homogenous magnetic field, which for convenience we assume to be aligned with the x_3-axis,

$$\boldsymbol{B} = B\,\boldsymbol{e}_3\,, \qquad B\in\mathbb{R}\,, \qquad \boldsymbol{e}_3 := (0,0,1)^T\,. \tag{10}$$

Then from eq. (8) the components of the vector potential $\boldsymbol{A} = (A_1, A_2, A_3)^T$ are to be obtained via:

$$\boldsymbol{B} = \boldsymbol{D} \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ D_1 & D_2 & D_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = B\, \boldsymbol{e}_3. \tag{11}$$

For the construction of the associated vector potential let us first introduce the three quantities

$$Q_j(x_j) := \frac{x_j}{1+2\nu_j}\,(1+\nu_j-\nu_j R_j) = \frac{1}{1+2\nu_j}\,(1+\nu_j+\nu_j R_j)\,x_j\,, \tag{12}$$

which are nonlocal functions of the coordinate x_j and obviously obey the relations $D_j Q_j = 1$ for all $j = 1, 2, 3$. With this in mind it is now straightforward to obtain a one-parameter family of vector potentials all resulting in the same magnetic field (10).

$$A_1^{(a)} = a\,BQ_2(x_2), \quad A_2^{(a)} = (1+a)\,Q_1(x_1), \quad A_3^{(a)} = 0\,, \tag{13}$$

where $a \in \mathbb{R}$ is a free gauge-parameter. As the Q's are nonlocal functions so are the components of $\boldsymbol{A}$ not simple functions but nonlocal functions as they depend on the R's. For convenience, let us specify some special cases.

Case $a = -\frac{1}{2}$:

$$A_1^{(-1/2)} = -\frac{B}{2}Q_2\,, \quad A_2^{(-1/2)} = \frac{B}{2}Q_1\,, \quad A_3^{(-1/2)} = 0 \tag{14}$$

Case $a = -1$**:**

$$A_1^{(-1)} = -BQ_2\,, \quad A_2^{(-1)} = 0\,, \quad A_3^{(-1)} = 0 \tag{15}$$

Case $a = 0$**:**

$$A_1^{(0)} = 0\,, \quad A_2^{(0)} = BQ_1\,, \quad A_3^{(0)} = 0. \tag{16}$$

One of the most important problems in physics is the Landau problem, which describes the properties of a charged particle in a constant magnetic field. In view of the results of refs. [31, 32], which investigated the Landau problem in commutative and non-commutative space, we can interpret eqs. (14-16) such that (14) represents a generalization of the symmetric gauge for the Landau problem, while eqs. (15) and (16) are analogs of the Landau gauge. In other words, in current manuscript presents the generalized Landau problem for the case of "Wigner-Dunkl quantum mechanics".

Apparently, two different gauge parameters, say a and b, result in the same constant magnetic field. Therefore, a gauge transformation must exist which transforms between two different but equivalent vector potentials as follows

$$\boldsymbol{A}^{(a)} \to \boldsymbol{A}^{(b)} := \boldsymbol{A}^{(a)} + \boldsymbol{D}\,\Lambda^{(b-a)} \tag{17}$$

where $\Lambda^{(b-a)}$ directly follows from the condition

$$\boldsymbol{A}^{(b)} - \boldsymbol{A}^{(a)} = \boldsymbol{D}\,\Lambda^{(b-a)}\,. \tag{18}$$

Again with $D_j Q_j = 1$ it is straightforward to show that this function is given by

$$\Lambda^{(b-a)} := (b-a)BQ_1Q_2\,. \tag{19}$$

4 Towards a gauge invariance of Wigner-Dunkl quantum mechanics

To begin with let us promote the above nonlocal functions (12) to nonlocal operators by considering the variable x_j in (12) as position operator acting on $\mathcal{H}$, that is, $Q_j = Q_j(X_j)$. They obviously obey the commutation relations

$$[D_i, Q_j] = \delta_{ij} \qquad \text{for all} \qquad i, j = 1, 2, 3, \tag{20}$$

and hence, may be called Dunkl position operators. It must be noted that these operators are not self-adjoint on $\mathcal{H}$ as obviously

$$Q_j^\dagger = \frac{x_j}{1+2\nu_j}(1+\nu_j+\nu_j R_j) = \frac{1}{1+2\nu_j}(1+\nu_j-\nu_j R_j)x_j\,. \tag{21}$$

We also note that $Q_j x_j = x_j Q_j^\dagger$ and hence

$$Q_j x_j^{2n} = x_j^{2n} Q_j\,, \qquad Q_j x_j^{2n+1} = x_j^{2n+1} Q_j^\dagger\,, \qquad \text{for} \qquad n \in \mathbb{Z}\,. \tag{22}$$

In the same way we will promote the gauge function $\Lambda^{(d)}$ as defined in (19) to a nonlocal operator and note that

$$[D_1, \Lambda^{(d)}] = dBQ_2\,, \qquad [D_2, \Lambda^{(d)}] = dBQ_1\,, \qquad [D_3, \Lambda^{(d)}] = 0\,, \tag{23}$$

which implies $[D_1,[D_1,\Lambda^{(d)}]] = 0$ and $[D_2,[D_2,\Lambda^{(d)}]] = 0$. Hence we may apply Hadamard's lemma leading us to the relations

$$\exp\left\{\frac{\mathrm{i}e}{\hbar c}\Lambda^{(d)}\right\} D_1 \exp\left\{-\frac{\mathrm{i}e}{\hbar c}\Lambda^{(d)}\right\} = D_1 - \frac{\mathrm{i}e}{\hbar c}[D_1,\Lambda^{(d)}] = D_1 - \frac{\mathrm{i}e}{\hbar c}dBQ_2\,, \tag{24}$$

$$\exp\left\{\frac{\mathrm{i}e}{\hbar c}\Lambda^{(d)}\right\} D_2 \exp\left\{-\frac{\mathrm{i}e}{\hbar c}\Lambda^{(d)}\right\} = D_2 - \frac{\mathrm{i}e}{\hbar c}[D_2,\Lambda^{(d)}] = D_2 - \frac{\mathrm{i}e}{\hbar c}dBQ_1\,. \tag{25}$$

This then results in

$$\exp\left\{\frac{\mathrm{i}e}{\hbar c}\Lambda^{(b-a)}\right\}\left(\frac{\hbar}{\mathrm{i}}\boldsymbol{D} - \frac{e}{c}\boldsymbol{A}^{(a)}\right)\exp\left\{-\frac{\mathrm{i}e}{\hbar c}\Lambda^{(b-a)}\right\} = \left(\frac{\hbar}{\mathrm{i}}\boldsymbol{D} - \frac{e}{c}\boldsymbol{A}^{(b)}\right)\,. \tag{26}$$

We are now in a position to establish the gauge invariance for Wigner-Dunkl quantum mechanics for a constant magnetic field. Let

$$H^{(a)} := \frac{1}{2m}\left(\boldsymbol{P} - \frac{e}{c}\boldsymbol{A}^{(a)}\right)^2 \qquad \text{and} \qquad H^{(b)} := \frac{1}{2m}\left(\boldsymbol{P} - \frac{e}{c}\boldsymbol{A}^{(b)}\right)^2\,, \tag{27}$$

then these two Hamiltonians are related to each other by the gauge transformation

$$\mathcal{G} := \exp\left\{\frac{\mathrm{i}e}{\hbar c}\Lambda^{(b-a)}\right\} \tag{28}$$

as follows

$$\mathcal{G}H^{(a)}\mathcal{G}^{-1} = H^{(b)}\,. \tag{29}$$

The corresponding states transform accordingly as

$$\psi^{(b)} = \mathcal{G}\psi^{(a)}\,. \tag{30}$$

Here we note that $Q_j^\dagger \neq Q_j$ and hence, also the generator (19) of gauge transformations is not self-adjoint. This implies that these gauge transformations are not unitary. We will investigate the transformation (30) of wave functions in more detail in the next section.

5 On the gauge transformation of wave functions

In this section, for convenience, we ignore the x_3 dependency, which is no limitation as the third component of the vector potential vanishes for all gauge parameters a in (13). Hence, we consider an arbitrary element $\psi \in \mathcal{H}$ and define its even and odd parts with respect to x_1 and x_2 by

$$\psi_{r_1 r_2}(x_1, x_2) := \frac{1}{4}\left[\psi(x_1, x_2) + r_1\psi(-x_1, x_2) + r_2\psi(x_1, -x_2) + r_1 r_2\psi(-x_1, -x_2)\right] . \tag{31}$$

These parts are pairwise orthogonal to each other and are eigenfunction of reflection operators R_1 and R_2.

$$R_i\psi_{r_1 r_2}(x_1, x_2) = r_i\psi_{r_1 r_2}(x_1, x_2) , \qquad r_i = \pm 1 , \quad i = 1, 2 . \tag{32}$$

The complete set of states $\psi_{r_1 r_2}$ for a fixed tuple (r_1, r_2) span the subspace $\mathcal{H}_{r_1 r_2} \in \mathcal{H}$ with well-defined parity corresponding to the eigenvalues r_1 and r_2 of the reflection operators R_1 and R_2, respectively. In other words, we have the orthogonal decomposition $\mathcal{H} = \mathcal{H}_{++} \oplus \mathcal{H}_{+-} \oplus \mathcal{H}_{-+} \oplus \mathcal{H}_{--}$ of the Hilbert space into the four eigen-subspaces of R_1 and R_2. The original wave function is recovered via

$$\psi(x_1, x_2) = \psi_{++}(x_1, x_2) + \psi_{+-}(x_1, x_2) + \psi_{-+}(x_1, x_2) + \psi_{--}(x_1, x_2) . \tag{33}$$

Here and below we will use a simplified notation for subindexes $r_i = \pm$. To investigate the action of the gauge operator we first consider the action of the components Q_j on the four components of the wave function:

$$Q_1\psi_{+\pm} = \frac{x_1}{1 + 2\nu_1}\psi_{+\pm} , \qquad Q_1\psi_{-\pm} = x_1\psi_{-\pm} , \tag{34}$$

$$Q_2\psi_{\pm+} = \frac{x_2}{1 + 2\nu_2}\psi_{\pm+} , \qquad Q_2\psi_{\pm-} = x_2\psi_{\pm-} . \tag{35}$$

The action of the adjoint operators on these components read for all eight cases

$$Q_1^\dagger\psi_{+\pm} = x_1\psi_{+\pm} , \qquad Q_1^\dagger\psi_{-\pm} = \frac{x_1}{1 + 2\nu_1}\psi_{-\pm} , \tag{36}$$

$$Q_2^\dagger\psi_{\pm+} = x_2\psi_{\pm+} , \qquad Q_2^\dagger\psi_{\pm-} = \frac{x_2}{1 + 2\nu_2}\psi_{\pm-} . \tag{37}$$

Combinations of these relations lead us to

$$Q_1 Q_2\psi = x_1 x_2\left[\frac{\psi_{++}}{(1 + 2\nu_1)(1 + 2\nu_2)} + \frac{\psi_{+-}}{(1 + 2\nu_1)} + \frac{\psi_{-+}}{(1 + 2\nu_2)} + \psi_{--}\right] , \tag{38}$$

$$Q_1^\dagger Q_2^\dagger\psi = x_1 x_2\left[\psi_{++} + \frac{\psi_{+-}}{(1 + 2\nu_2)} + \frac{\psi_{-+}}{(1 + 2\nu_1)} + \frac{\psi_{--}}{(1 + 2\nu_1)(1 + 2\nu_2)}\right] , \tag{39}$$

and we observe, using $Q_1 Q_2 x_1 x_2 = x_1 x_2 Q_1^\dagger Q_2^\dagger$, that

$$(Q_1 Q_2)^2\,\psi = q_1^2 q_2^2\psi \qquad \text{with} \qquad q_j := \frac{x_j}{1 + 2\nu_j} \tag{40}$$

This finally leads us to the general gauge transformation of wave functions

$$\exp\{-\mathrm{i}\alpha Q_1 Q_2\}\,\psi = \cos(\alpha q_1 q_2)\psi - \mathrm{i}\sin(\alpha q_1 q_2)\frac{1}{q_1 q_2}Q_1 Q_2\psi . \tag{41}$$

where $\alpha = (b - a)eB/(\hbar c)$. For the components of the wave function within the subspace $\mathcal{H}_{r_1 r_2}$ this reads

$$\mathcal{G}\psi_{r_1 r_2} = \left[\cos(\alpha q_1 q_2) - \mathrm{i}\sin(\alpha q_1 q_2)(1 + 2\nu_1)^{\frac{1-r_1}{2}}(1 + 2\nu_2)^{\frac{1-r_2}{2}}\right]\psi_{r_1 r_2} . \tag{42}$$

Or more explicitly we have

$$\begin{aligned}
\mathcal{G}\psi_{++} &= \exp\{-\mathrm{i}\alpha q_1 q_2\}\,\psi_{++} , \\
\mathcal{G}\psi_{+-} &= [\cos(\alpha q_1 q_2) - \mathrm{i}(1 + 2\nu_2)\sin(\alpha q_1 q_2)]\,\psi_{+-} , \\
\mathcal{G}\psi_{-+} &= [\cos(\alpha q_1 q_2) - \mathrm{i}(1 + 2\nu_1)\sin(\alpha q_1 q_2)]\,\psi_{-+} , \\
\mathcal{G}\psi_{--} &= [\cos(\alpha q_1 q_2) - \mathrm{i}(1 + 2\nu_1)(1 + 2\nu_2)\sin(\alpha q_1 q_2)]\,\psi_{--} .
\end{aligned} \tag{43}$$

Obviously, the norm is only preserved for the symmetric component ψ_{++}. In general, $\mathcal{G}$ will only be unitarity in the limit where both, ν_1 and ν_2, vanish. In that sense we may view $\mathcal{G}$ as an operator with deformed unitary as

$$\begin{aligned}
|\mathcal{G}\psi_{++}|^2 &= |\psi_{++}|^2 \,,\\
|\mathcal{G}\psi_{+-}|^2 &= \left(\cos^2(\alpha q_1 q_2) + \sin^2(\alpha q_1 q_2)(1+2\nu_2)^2\right) |\psi_{+-}|^2 \,,\\
|\mathcal{G}\psi_{-+}|^2 &= \left(\cos^2(\alpha q_1 q_2) + \sin^2(\alpha q_1 q_2)(1+2\nu_1)^2\right) |\psi_{-+}|^2 \,,\\
|\mathcal{G}\psi_{--}|^2 &= \left(\cos^2(\alpha q_1 q_2) + \sin^2(\alpha q_1 q_2)(1+2\nu_1)^2(1+2\nu_2)^2\right) |\psi_{--}|^2 \,.
\end{aligned} \tag{44}$$

To conclude this section let us note that we have achieved gauge invariance of the Wigner-Dunkl quantum mechanics at the expense that unitarity is replaced by a deformed unitarity in above sense. Naturally the question arises, what happens in the case we minimally couple a standard vector potential to the Wigner-Dunkl Hamiltonian (4). This will be the subject of the next section.

6 On the gauge transformation for standard vector potentials

As mentioned above, this section is dedicated to the investigation of a minimal coupling of standard vector potentials to the free Wigner-Dunkl Hamiltonian (4). That is we will now consider the usual relation

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \tilde{\boldsymbol{A}} \,. \tag{45}$$

Here $\boldsymbol{\nabla}$ denotes the standard gradient operator and we will use the tilde symbol for standard quantities like vector potentials and gauge functions. Again we will consider the case of a homogenous magnetic field for simplicity and comparison with above findings. Obviously the general form of the vector potential generating such a magnetic field via (45) is given by

$$\tilde{A}_1^{(a)} = a\,Bx_2 , \quad \tilde{A}_2^{(a)} = (1+a)\,x_1 , \quad \tilde{A}_3^{(a)} = 0 \,, \tag{46}$$

and the gauge transformation reads

$$\tilde{\boldsymbol{A}}^{(a)} \to \tilde{\boldsymbol{A}}^{(b)} := \tilde{\boldsymbol{A}}^{(a)} + \boldsymbol{\nabla}\, \tilde{\Lambda}^{(b-a)} \quad \text{with} \quad \tilde{\Lambda}^{(b-a)} := (b-a)Bx_1x_2 \,. \tag{47}$$

Let us now minimally couple this vector potential to the free Wigner-Dunkl Hamiltonian. In doing so we consider the unitary operator

$$\tilde{\mathcal{G}} := \exp\left\{\frac{\mathrm{i}e}{\hbar c}\tilde{\Lambda}^{(b-a)}\right\} = \exp\left\{\frac{\mathrm{i}eB}{\hbar c}(b-a)x_1x_2\right\} \tag{48}$$

and observe that

$$\tilde{\mathcal{G}}R_j = R_j\tilde{\mathcal{G}}^\dagger \qquad \text{for} \qquad j = 1,2 \,, \tag{49}$$

due to the fact $\tilde{\Lambda}^{(b-a)}$ is an odd function in x_1 and x_2. This implies that we cannot establish a gauge invariance for the Wigner-Dunkl Hamiltonian minimally coupled to a standard vector potential as

$$\tilde{\mathcal{G}}\left(D_j - \frac{\mathrm{i}e}{\hbar c}\tilde{A}_j^{(a)}\right)\tilde{\mathcal{G}}^{-1} = \left(D_j - \frac{\mathrm{i}e}{\hbar c}\tilde{A}_j^{(b)}\right) + \frac{\nu_j}{x_j}\left(R_j - \mathcal{G}R_j\mathcal{G}^\dagger\right) \,. \tag{50}$$

7 Summary

For Wigner-Dunkl quantum mechanics being a physically self-contained deformed quantum system it is essential that it exhibits certain basic features. One of these basic properties is gauge invariance. By generalizing the Landau problem for the case of “Wigner-Dunkl quantum mechanics”, we have shown that this can be achieved to a certain extend when using the Dunkl-Maxwell formalism. That is, the spectra of the two gauge-equivalent

Hamiltonians in (29) are identical. This comes at the expense that the gauge-transformation is not unitary and hence the probability densities are not invariant under such transformation but transform according to (44).

On the other hand, we could show that, when coupled to a standard vector potential $\tilde{\boldsymbol{A}}$, it does not lead to any meaningful quantum system as different gauges would in general result in different spectra as indicated by (50). The current discussion is limited to the case of a constant magnetic field and, therefore, can only be a first step towards a gauge invariant Wigner-Dunkl quantum formalism. Further investigations in that direction are currently in progress. For example, the usual Landau problem is characterized by four integrals of motion that generate the centrally extended $e(2)\oplus u(1)$ algebra. Naturally one might ask the question, what happens to this symmetry within the Wigner-Dunkl formalism. See for a similar discussion ref. [33], where the noncommutative Landau problem was investigated in an extended supersymmetric context. It is well-known, see for example [34], that the Pauli equation for a charged spin-$\frac{1}{2}$ particle exhibits a supersymmetry for gyromagnetic ratio 2. This is also expected to emerge in the limit of vanishing deformation parameters $\nu_i = 0$ but a more detailed investigation is required. Similarly, the usual Landau problem is an exactly solvable system with the Hamiltonian operator being in essence equivalent to the Hamiltonian of the one-dimensional oscillator. Hence, in the limit of vanishing deformation parameters in the Wigner-Dunkl formalism, the energy eigenvalues of the usual oscillator with its infinite degeneracy, i.e. the so-called Landau levels, shall emerge. This, however, also requires a more detailed discussion.

Let us also briefly comment on the Dunkl-position operators introduced in section 4. In contrast to the Dunkl momentum operators P_j being Hermitian, the Q_j's are not and, hence, may not be of much use beyond the current context. This deficit can be cured when looking at symmetrized quantities like

$$\frac{1}{2}\left(Q_j + Q_j^\dagger\right) = x_j\,\frac{1+\nu_j}{1+2\nu_j} \qquad \text{or} \qquad \frac{1}{2}\left\{Q_j, Q_j^\dagger\right\} = x_j^2\,\frac{1+2\nu_j+2\nu_j^2}{(1+2\nu_j)^2}\,, \tag{51}$$

which are obviously self-adjoined and local operators.

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Appendix: Hermitianness of P_j

The Hermitianness of the components P_j requires that $(\varphi, P_j\psi) = (P_j\varphi, \psi)$ for all $\varphi, \psi \in \mathcal{H}$. Hence it is sufficient to show that the Dunkl operators D_j obey $(\varphi, D_j\psi) = -(D_j\varphi, \psi)$. First, by looking only on a single component $x_j = x$, we note that

$$(\varphi, D_j\psi) \;\; = \int_{\mathbb{R}} dx\,|x|^{2\nu}\,\left(\varphi^*(x)\psi'(x) + \frac{\nu}{x}\varphi^*(x)\psi(x) - \frac{\nu}{x}\varphi^*(x)\psi(-x)\right)\,. \tag{52}$$

Let us now consider

$$(D_j\varphi, \psi) \;\; = \int_{\mathbb{R}} dx\,|x|^{2\nu}\,\left(\varphi'^*(x)\psi(x) + \frac{\nu}{x}\varphi^*(x)\psi(x) - \frac{\nu}{x}\varphi^*(-x)\psi(x)\right)\,. \tag{53}$$

Integration by parts with $\partial_x(|x|^{2\nu}\psi(x)) = |x|^{2\nu}\left(\psi'(x)) + \frac{2\nu}{x}\psi(x)\right)$ results in

$$\begin{aligned}(\varphi, D_j\psi) \;\; &= \int_{\mathbb{R}} dx\,|x|^{2\nu}\,\left(-\varphi^*(x)\psi'(x) - \frac{\nu}{x}\varphi^*(x)\psi(x) - \frac{\nu}{x}\varphi^*(-x)\psi(x)\right)\\ &= -(\varphi, D_j\psi)\,,\end{aligned} \tag{54}$$

which completes the proof for all components P_j being Hermitian.

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