

Exercise 11: Wigner's construction of left-invariant measure for Lie groups

Assume a left-invariant measure exists: $d\mu(g) = d\mu(g_0^{-1}g)$

Then $g = g(\alpha_1, \dots, \alpha_n)$ and, with g_0 fixed, $g_0^{-1}g = g_1 = g(\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon = \varepsilon(\alpha)$

Invariance implies

$$\rho(\alpha_1, \dots, \alpha_n) d^n \alpha = \rho(\varepsilon_1, \dots, \varepsilon_n) d^n \varepsilon$$

$\Rightarrow$

$$\rho(\varepsilon_1, \dots, \varepsilon_n) = \rho(\alpha_1, \dots, \alpha_n) \frac{\partial(\varepsilon_1, \dots, \varepsilon_n)}{\partial(\alpha_1, \dots, \alpha_n)}$$

Let $P_k(g) := \alpha_k$ function which provides the k -th parameter of group element g

Idea: Let ε_i be very small, that is, $g_1 = g(\varepsilon_1, \dots, \varepsilon_n) \sim e$ is very close to neutral element

Then $g = g_0 g_1 = g(\alpha)g(\varepsilon)$

$$\boxed{\rho(\alpha_1, \dots, \alpha_n) = \rho_0 \lim_{\varepsilon \rightarrow 0} \left[\frac{\partial(P_1(g(\alpha)g(\varepsilon)), \dots, P_n(g(\alpha)g(\varepsilon)))}{\partial(\varepsilon_1, \dots, \varepsilon_n)} \right]^{-1}}$$

Problem is reduced to expand functions P_k for small ε

$\rho_0 = \rho(0, \dots, 0)$ is density of matrix elements near neutral element $\Leftrightarrow$ normalisation

Example: $G = SO(2)$

$$g = g(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \quad g = g(\varepsilon) = \begin{pmatrix} \cos \varepsilon & -\sin \varepsilon \\ \sin \varepsilon & \cos \varepsilon \end{pmatrix}$$

$$P(g(\alpha)g(\varepsilon)) = P(g(\alpha + \varepsilon)) = \alpha + \varepsilon \quad \Rightarrow \quad \frac{\partial P}{\partial \varepsilon} = 1$$

$$\Rightarrow \quad \rho(\alpha) = \rho_0 := \frac{1}{2\pi} \quad \Rightarrow \quad d\mu(g) = \frac{1}{2\pi} d\alpha$$

In general and in particular for non-abelian groups the problem is to find an explicit form for functions $P_k(g(\alpha)g(\varepsilon))$.

Exercise 12: Stanley's n -vector model on the linear chain

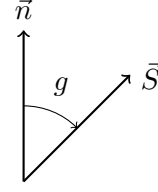
"Spin":

$\vec{S} \in S^{n-1}$ "spin space"

$SO(n)$ transitive transformation group

Let $\vec{S} = g\vec{n}$, $\vec{n}$ = north pole, $g \in SO(n)$

$SO(n-1) = \{g \in SO(n) | g\vec{n} = \vec{n}\}$ massive



Invariant Haar measure dg : $\int_{SO(n)} dg = 1$

Invariant normalised measure on S^{n-1} :

$$d\mu(\vec{S}) = \frac{\Gamma(n/2)}{2\pi^{n/2}} d^{n-1}\Omega(\vec{S})$$

is a priori normalised probability distribution of $\vec{S}$

"Spin interaction":

Shall be $SO(n)$ invariant and exchange invariant

$$V(\vec{S}, \vec{S}') = V(g\vec{S}, g\vec{S}') = V(\vec{S}', \vec{S})$$

For example: $V(\vec{S}, \vec{S}') = -J\vec{S} \cdot \vec{S}' - K(\vec{S} \cdot \vec{S}')^2$

$K = 0$: Stanley's n -vector model

$K > 0$: Stanley's n -vector model on harmonic chain

Consider:

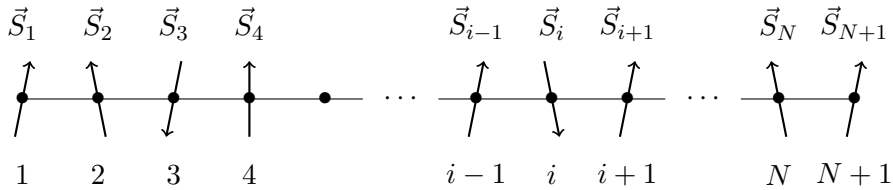
$$v(g) := V(g\vec{n}, \vec{n}) = V(\vec{n}, g\vec{n}) = v(g^{-1})$$

is a zonal spherical function as for all $h, h' \in SO(n-1)$

$$v(hgh') = V(hgh'\vec{n}, \vec{n}) = V(hg\vec{n}, \vec{n}) = V(g\vec{n}, h^{-1}\vec{n}) = V(g\vec{n}, \vec{n}) = v(g)$$

Linear chain:

Chain of $N+1$ sites, each occupied by a "spin" $\vec{S}_j := g_j\vec{n}$, $g_j \in SO(n)$



Nearest neighbour interaction:

$$H := \sum_{j=1}^N V(\vec{S}_j, \vec{S}_{j+1}) = \sum_{j=1}^N v(g_j^{-1}g_{j+1})$$

Partition function:

$$\begin{aligned} Z(\beta) &:= \int_{S^{n-1}} d\mu(\vec{S}_1) \cdots \int_{S^{n-1}} d\mu(\vec{S}_{N+1}) e^{-\beta H}, \quad \beta := 1/k_B T \\ &= \int_{SO(n)} dg_1 \cdots \int_{SO(n)} dg_{N+1} \prod_{j=1}^N \exp \left\{ -\beta v(g_j^{-1}g_{j+1}) \right\} \end{aligned}$$

Harmonic analysis:

Class 1 reps of $SO(n)$: $\ell \in \Lambda = \{0, 1, 2, 3, \dots\}$, $d_\ell = (2\ell + n - 2) \frac{\Gamma(\ell + n - 2)}{\Gamma(\ell + 1)\Gamma(n - 1)}$

$$L^2(S^{n-1}) = \sum_{\ell=0}^{\infty} \mathcal{H}^\ell, \quad \dim \mathcal{H}^\ell = d_\ell$$

As $v(g)$ is zonal spherical function

$$\begin{aligned} e^{-\beta v(g)} &= \sum_{\ell=0}^{\infty} d_\ell \lambda_\ell(\beta) D_{00}^\ell(g) \\ \lambda_\ell(\beta) &= \int_{SO(n)} dg e^{-\beta v(g)} D_{00}^{\ell*}(g) \end{aligned}$$

Hence

$$\begin{aligned} Z(\beta) &= \int_{SO(n)} dg_1 \cdots \int_{SO(n)} dg_{N+1} \sum_{\ell_1=0}^{\infty} \cdots \sum_{\ell_N=0}^{\infty} d_{\ell_1} \cdots d_{\ell_N} \lambda_{\ell_1}(\beta) \cdots \lambda_{\ell_N}(\beta) \\ &\quad \times D_{00}^{\ell_1}(g_1^{-1}g_2) \cdots D_{00}^{\ell_N}(g_N^{-1}g_{N+1}) \end{aligned}$$

Orthogonality:

$$\begin{aligned} &\int_{SO(n)} dg_j D_{00}^{\ell_{j-1}}(g_{j-1}^{-1}g_j) D_{00}^{\ell_j}(g_j^{-1}g_{j+1}) \\ &= \sum_{mn} D_{0m}^{\ell_{j-1}}(g_{j-1}^{-1}) \underbrace{\int_{SO(n)} dg_j D_{m0}^{\ell_{j-1}}(g_j) D_{n0}^{\ell_j*}(g_j) D_{00}^{\ell_j}(g_{j+1})}_{=\delta_{\ell_{j-1}\ell_j} \delta_{mn} / d_{\ell_j}} \\ &= \frac{\delta_{\ell_{j-1}\ell_j}}{d_{\ell_j}} \sum_m D_{0m}^{\ell_{j-1}}(g_{j-1}^{-1}) D_{m0}^{\ell_j}(g_{j+1}) \\ &= \frac{\delta_{\ell_{j-1}\ell_j}}{d_{\ell_j}} D_{00}^{\ell_j}(g_{j-1}^{-1}g_{j+1}) \end{aligned}$$

$\Rightarrow$

$$Z(\beta) = \int_{SO(n)} dg_1 \int_{SO(n)} dg_{N+1} \sum_{\ell=0}^{\infty} d_\ell [\lambda_\ell(\beta)]^N D_{00}^\ell(g_1^{-1}g_{N+1})$$

For open boundaries:

$$\begin{aligned} \int_{SO(n)} dg_{N+1} D_{00}^\ell(g_1^{-1}g_{N+1}) &= \int_{SO(n)} dg D_{00}^\ell(g) = \delta_{\ell 0} \\ \int_{SO(n)} dg_1 &= 1 \end{aligned}$$

Result:

$$\boxed{Z(\beta) = \lambda_0^N(\beta)}$$

with

$$\lambda_0(\beta) = \int_{SO(n)} dg e^{-\beta v(g)} = \int_{S^{n-1}} d\mu(\vec{S}) \exp\{-\beta V(\vec{S}, \vec{n})\}$$

For fixed boundaries: $\vec{S}_1 = g_1 \vec{n}$ and $\vec{S}_{N+1} = g_{N+1} \vec{n}$

$$\begin{aligned} Z_{\text{fix}}(\beta) &= \sum_{\ell=0}^{\infty} d_\ell [\lambda_\ell(\beta)]^N D_{00}^\ell(g_1^{-1}g_{N+1}) \\ &= Z(\beta) \left(1 + \sum_{\ell \neq 0} d_\ell \left[\frac{\lambda_\ell(\beta)}{\lambda_0(\beta)} \right]^N D_{00}^\ell(g_1^{-1}g_{N+1}) \right) \end{aligned}$$

For periodic boundaries: $\vec{S}_1 = \vec{S}_{N+1} \Rightarrow g_1 = g_{N+1} \Rightarrow D_{00}^\ell(g_1^{-1}g_{N+1}) = D_{00}^\ell(e) = 1$

$$Z_{\text{fix}}(\beta) = Z(\beta) \left(1 + \sum_{\ell \neq 0}^\infty d_\ell \left[\frac{\lambda_\ell(\beta)}{\lambda_0(\beta)} \right]^N \right)$$

Large N limit:

Unitarity: $|D_{00}^\ell(g)| \leq D_{00}^0(g) = 1$

For $\ell \neq 0$ we may conclude $|\lambda_\ell(\beta)| < \int dg e^{-\beta v(g)} = \lambda_0(\beta)$ for all $\beta \Rightarrow \left| \frac{\lambda_\ell(\beta)}{\lambda_0(\beta)} \right| < 1$

Free energy:

$$F(\beta) := -\frac{1}{\beta} \lim_{N \rightarrow \infty} \frac{1}{N+1} \ln Z(\beta) = -\frac{1}{\beta} \ln \lambda_0(\beta)$$

Result is valid for all classical spin models with "spin space" = G/H with H massive and G -invariant and exchange invariant spin-spin interaction between nearest neighbours.

Explicit results for n -vector model

Assume $V(\vec{S}, \vec{S}') = f(\vec{S} \cdot \vec{S}')$, which is most general form of invariant interaction
 $\Rightarrow V(g\vec{n}, \vec{n}) = v(g) = f(\cos \theta)$

$$D_{00}^\ell(g) = \frac{\Gamma(\ell+1)\Gamma(n-2)}{\Gamma(\ell+n-2)} C_\ell^{\frac{n-2}{2}}(\cos \theta), \quad D_{00}^1(g) = \cos \theta = g\vec{n} \cdot \vec{n}$$

$$\begin{aligned} \lambda_\ell(\beta) &= \int_{SO(n)} dg e^{-\beta v(g)} D_{00}^\ell(g) \\ &= \frac{\Gamma(n/2)\Gamma(\ell+1)\Gamma(n-2)}{\sqrt{\pi}\Gamma(\frac{n-2}{2})\Gamma(\ell+n-2)} \int_0^\pi d\theta \sin^{n-2} \theta e^{-\beta f(\cos \theta)} C_\ell^{\frac{n-2}{2}}(\cos \theta) \end{aligned}$$

In particular:

$$\begin{aligned} \lambda_0(\beta) &= \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_{-1}^1 dt e^{-\beta f(t)} (1-t^2)^{\frac{n-3}{2}} \\ \lambda_1(\beta) &= \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_{-1}^1 dt e^{-\beta f(t)} t(1-t^2)^{\frac{n-3}{2}} \end{aligned}$$

More results:

Free energy: $F(\beta) = -\frac{1}{\beta} \ln \lambda_0(\beta)$

Internal energy: $\frac{\partial}{\partial \beta} (\beta F(\beta)) = -\lambda_0(\beta)/\lambda_0'(\beta)$

Entropy: $S(\beta) = k_B \beta (E(\beta) - F(\beta)) = k_B \left(\ln \lambda_0(\beta) - \beta \frac{\lambda_0'(\beta)}{\lambda_0(\beta)} \right)$

Heat capacity: $c(\beta) = -k_B \beta^2 \frac{\partial E(\beta)}{\partial \beta} = k_B \beta^2 \left[\frac{\lambda_0''(\beta)}{\lambda_0(\beta)} - \left(\frac{\lambda_0'(\beta)}{\lambda_0(\beta)} \right)^2 \right]$

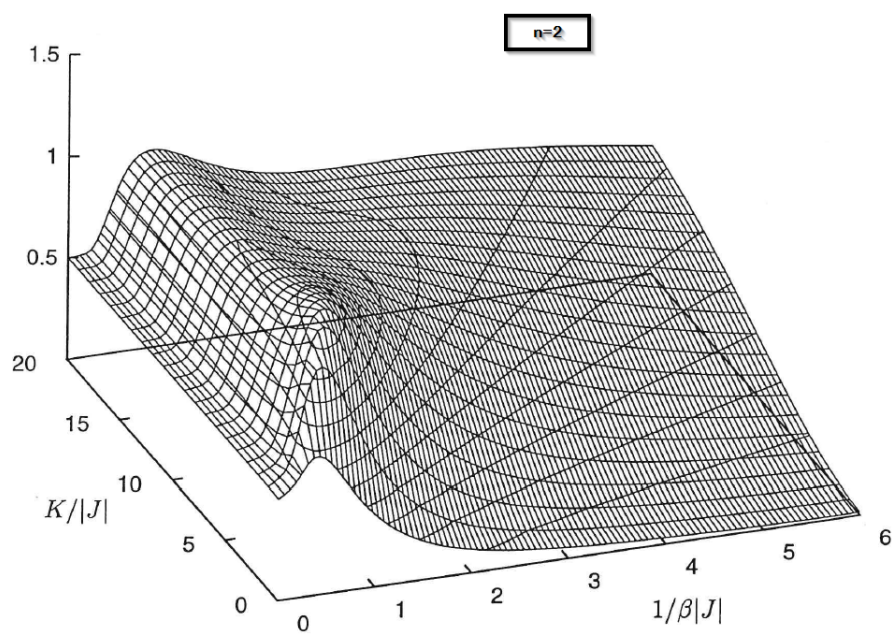
Spin correlation: $\langle \vec{S}_i \cdot \vec{S}_{i+r} \rangle = \langle D_{00}^1(g_i^{-1}g_{i+r}) \rangle = \left(\frac{\lambda_1(\beta)}{\lambda_0(\beta)} \right)^r$

Zero-field susceptibility: $\chi_0(\beta) = \beta \left(1 + 2 \sum_{r=1}^\infty \langle \vec{S}_i \cdot \vec{S}_{i+r} \rangle \right) = \beta \frac{1 + \frac{\lambda_1(\beta)}{\lambda_0(\beta)}}{1 - \frac{\lambda_1(\beta)}{\lambda_0(\beta)}}$

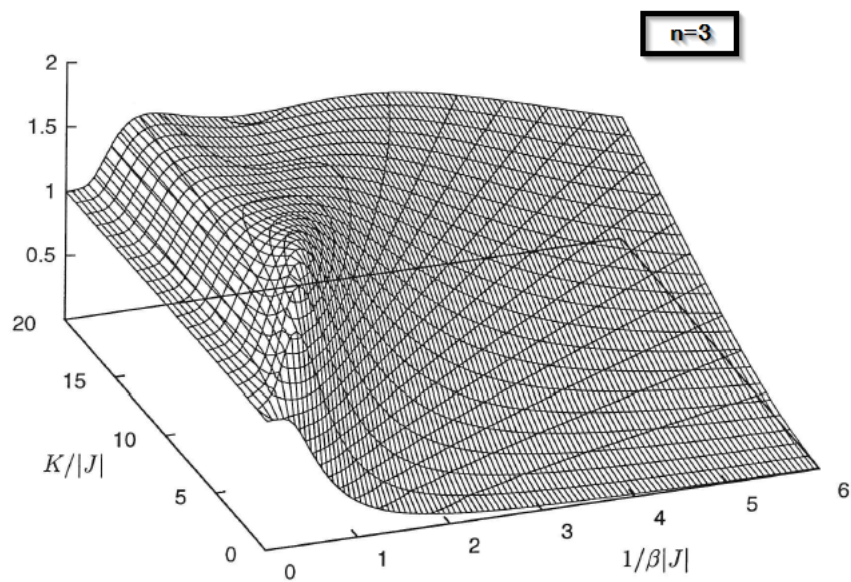
Next pages explicit results for harmonic chain with $V(\vec{S}, \vec{S}') = -J\vec{S} \cdot \vec{S}' - K(\vec{S} \cdot \vec{S}')^2$

and "Kicked Rotator" (classical and quantum chaos)

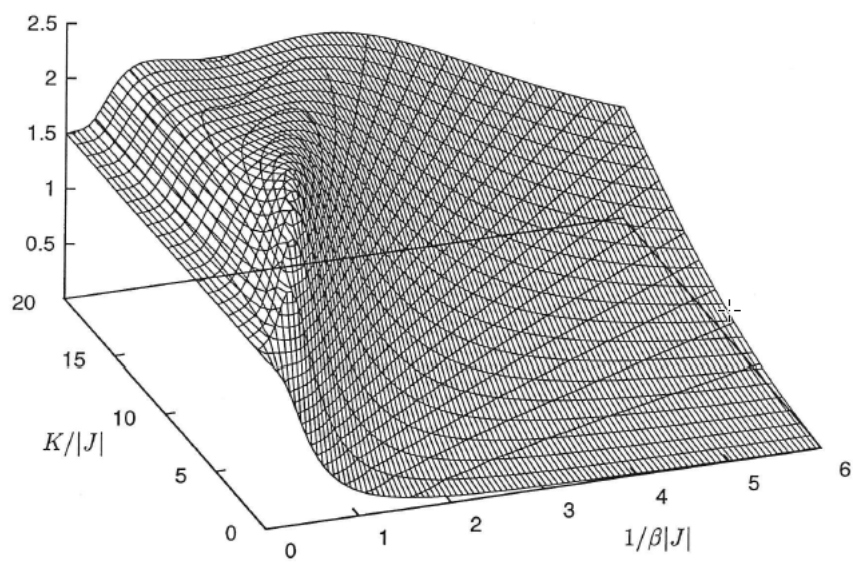
$C(\beta)/k_B$



$C(\beta)/k_B$



$C(\beta)/k_B$



Exercise 13: Wigner polynomials

Consider d_j dimensional UIR of $SU(2)$ via Euler angles acting in $\mathbb{C}^{2j+1}$

$$D^{(j)}(\varphi, \theta, \psi) := e^{-i\varphi J_3} e^{-i\theta J_2} e^{-i\psi J_3}$$

In standard basis: $J_3|jm\rangle = m|jm\rangle$ with $(J_1^2 + J_2^2 + J_3^2)|jm\rangle = j(j+1)|jm\rangle$
Matrix elements of D^j :

$$\langle jm|D^{(j)}(\varphi, \theta, \psi)|jn\rangle = e^{-im\varphi} d_{mn}^{(j)}(\theta) e^{-in\psi}$$

with the *Wigner polynomials*

$$d_{m,n}^{(j)}(\theta) := \langle jm|e^{-i\theta J_2}|jn\rangle$$

Problem: Find explicit form of Wigner polynomials

Solution:

$$\begin{aligned} d_{m,n}^{(j)}(\theta) &= \sqrt{\binom{j+m}{j-n} \binom{j-n}{j-m}} \cos^{m+n} \frac{\theta}{2} \sin^{m-n} \frac{\theta}{2} {}_2F_1(-j+m, j+m+1; m-n; \sin^2 \frac{\theta}{2}) \\ &= \sqrt{\frac{(j+m)!(j-m)!}{(j+n)!(j-n)!}} \cos^{m+n} \frac{\theta}{2} \sin^{m-n} \frac{\theta}{2} P_{j-m}^{(m-n, m+n)}(\cos \theta) \end{aligned}$$

${}_2F_1(a, b; c; z)$ is the hypergeometric function

$P_l^{(m,n)}(z)$ is Jacobi polynomial

Properties:

- $d_{m,n}^{(j)}(\theta) = (-1)^{m-n} d_{-m,-n}^{(j)}(\theta) = (-1)^{m-n} d_{n,m}^{(j)}(\theta)$
- $d_{m,0}^{(j)}(\theta) = (-1)^m d_{0,m}^{(j)}(\theta) = \sqrt{\frac{(j-m)!}{(j+m)!}} P_j^m(\cos \theta)$ ass. Legendre Polynomial
- $d_{0,0}^{(j)}(\theta) = P_j(\cos \theta)$ Legendre Polynomial
- $Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}} d_{m,0}^{(l)}(\theta) e^{im\varphi}$ here $l = 0, 1, 2, 3, \dots$

Explicitly: In order $(\frac{1}{2}, -\frac{1}{2})$ and $(1, 0-1)$ from left to right and top to bottom

$$\begin{aligned} d^{(1/2)}(\theta) &= \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \\ d^{(1)}(\theta) &= \begin{pmatrix} \cos^2 \frac{\theta}{2} & -\frac{1}{\sqrt{2}} \sin \theta & \sin^2 \frac{\theta}{2} \\ \frac{1}{\sqrt{2}} \sin \theta & \cos \theta & -\frac{1}{\sqrt{2}} \sin \theta \\ \sin^2 \frac{\theta}{2} & \frac{1}{\sqrt{2}} \sin \theta & \cos^2 \frac{\theta}{2} \end{pmatrix} \end{aligned}$$

More see: A. Lindner *Drehimpulse in der QM* (Teubner, Stuttgart, 1984)

Proof:

Step 1:

$$\frac{\partial^2}{\partial \theta^2} d_{m,n}^{(j)}(\theta) = -\langle jm | J_2^2 e^{-i\theta J_2} | jn \rangle$$

Step 2:

$$\begin{aligned} j(j+1)d_{m,n}^{(j)}(\theta) &= \langle jm | \vec{J}^2 e^{-i\theta J_2} | jn \rangle \\ &= \langle jm | (J_1^2 + J_2^2 + J_3^2) e^{-i\theta J_2} | jn \rangle \\ &= \langle jm | J_1^2 e^{-i\theta J_2} | jn \rangle - \frac{\partial^2}{\partial \theta^2} d_{m,n}^{(j)}(\theta) + m^2 d_{m,n}^{(j)}(\theta) \end{aligned}$$

Step 3:

Noting that (see Addendum Exercise 10)

$$\begin{aligned} e^{i\theta J_2} J_1^2 e^{-i\theta J_2} &= (J_1 \cos \theta + J_3 \sin \theta)^2 \\ &= J_1^2 \cos^2 \theta + J_3^2 \sin^2 \theta + (J_1 J_3 + J_3 J_1) \sin \theta \cos \theta \\ &= (\vec{J}^2 - J_2^2 - J_3^2) \cos^2 \theta + J_3^2 \sin^2 \theta + (2J_1 J_3 + iJ_2) \sin \theta \cos \theta \\ \langle jm | J_1^2 e^{-i\theta J_2} | jn \rangle &= \langle jm | e^{-i\theta J_2} e^{i\theta J_2} J_1^2 e^{-i\theta J_2} | jn \rangle \\ &= \langle jm | e^{-i\theta J_2} (\vec{J}^2 - J_2^2 - J_3^2) \cos^2 \theta | jn \rangle \\ &\quad + \langle jm | e^{-i\theta J_2} J_3^2 | jn \rangle \sin^2 \theta \\ &\quad + \langle jm | e^{-i\theta J_2} (2J_1 J_3 + iJ_2) | jn \rangle \sin \theta \cos \theta \\ &= \cos^2 \theta j(j+1)d_{m,n}^{(j)}(\theta) + \cos^2 \theta \frac{\partial^2}{\partial \theta^2} d_{m,n}^{(j)}(\theta) - n^2 \cos^2 \theta d_{m,n}^{(j)}(\theta) \\ &\quad + n^2 \sin^2 \theta d_{m,n}^{(j)}(\theta) \\ &\quad + 2n \sin \theta \cos \theta \langle jm | e^{-i\theta J_2} J_1 | jn \rangle - \sin \theta \cos \theta \frac{\partial}{\partial \theta} d_{m,n}^{(j)}(\theta) \end{aligned}$$

Step 4:

$$\begin{aligned} m d_{m,n}^{(j)}(\theta) &= \langle jm | J_3 e^{-i\theta J_2} | jn \rangle \\ &= \langle jm | e^{-i\theta J_2} e^{i\theta J_2} J_3 e^{-i\theta J_2} | jn \rangle \\ &= \langle jm | e^{-i\theta J_2} (J_3 \cos \theta - J_1 \sin \theta) | jn \rangle \\ &= n \cos \theta d_{m,n}^{(j)} - \sin \theta \langle jm | e^{-i\theta J_2} J_1 | jn \rangle \end{aligned}$$

Putting all together leads to a differential equation for the Wigner polynomial

$$\begin{aligned} j(j+1)d_{m,n}^{(j)}(\theta) &= -\frac{\partial^2}{\partial \theta^2} d_{m,n}^{(j)}(\theta) + m^2 d_{m,n}^{(j)}(\theta) \\ &\quad + \cos^2 \theta j(j+1)d_{m,n}^{(j)}(\theta) + \cos^2 \theta \frac{\partial^2}{\partial \theta^2} d_{m,n}^{(j)}(\theta) - n^2 \cos^2 \theta d_{m,n}^{(j)}(\theta) \\ &\quad + n^2 \sin^2 \theta d_{m,n}^{(j)}(\theta) - \sin \theta \cos \theta \frac{\partial}{\partial \theta} d_{m,n}^{(j)}(\theta) \\ &\quad + 2n \sin \theta \cos \theta \left(n \cos \theta d_{m,n}^{(j)} - m d_{m,n}^{(j)}(\theta) \right) \\ \left[j(j+1) \sin^2 \theta + \sin^2 \theta \frac{\partial^2}{\partial \theta^2} + \sin \theta \cos \theta \frac{\partial}{\partial \theta} - m^2 + n^2 \cos^2 \theta - n^2 \sin^2 \theta - 2n^2 \cos^2 \theta + 2nm \cos \theta \right] d_{m,n}^{(j)}(\theta) &= 0 \\ \left[\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} - \frac{m^2 + n^2 - 2nm \cos \theta}{\sin^2 \theta} + j(j+1) \right] d_{m,n}^{(j)}(\theta) &= 0 \\ \left[\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} - \frac{(\frac{m-n}{2})^2}{\sin^2 \frac{\theta}{2}} - \frac{(\frac{m+n}{2})^2}{\cos^2 \frac{\theta}{2}} + j(j+1) \right] d_{m,n}^{(j)}(\theta) &= 0 \end{aligned}$$

Hypergeometric differential eq. having Jacobi-polynomials as solution

See also QM problem for Pöschl-Teller potential

*** End of Tutorial 4 ***