

Exercise 7: Classes and Character table of D_4

The group table: $d^4 = e = s^2$, $d^k s = d^{4-k} s$

D_4	e	d	d^2	d^3	s	ds	$d^2 s$	$d^3 s$
e	e	d	d^2	d^3	s	ds	$d^2 s$	$d^3 s$
d	d	d^2	d^3	e	ds	$d^2 s$	$d^3 s$	s
d^2	d^2	d^3	e	d	$d^2 s$	$d^3 s$	s	ds
d^3	d^3	e	d	d^2	$d^3 s$	s	ds	$d^2 s$
s	s	$d^3 s$	$d^2 s$	ds	e	d^3	d^2	d
ds	ds	s	$d^3 s$	$d^2 s$	d	e	d^3	d^2
$d^2 s$	$d^2 s$	ds	s	$d^3 s$	d^2	d	e	d^3
$d^3 s$	$d^3 s$	$d^2 s$	ds	s	d^3	d^2	d	e

Center of D_4 : $Z(D_4) = \{e, d^2\} \Rightarrow$ each is a class by itself $\{e\}, \{d^2\}$

Consider element d : $\Rightarrow \{d, d^3\}$

Consider element s : $\Rightarrow \{s, d^2 s\}$

Remaining elements: $\Rightarrow \{ds, d^3 s\}$

Conclusion: D_4 has 5 classes given by

$$\{e\}, \{d^2\}, \{d, d^3\}, \{s, d^2 s\}, \{ds, d^3 s\}$$

For the general case D_n see Homework 3 Problem 6a)

For the character table D_4 see Homework 3 Problem 6a)

D_4	$\{e\}$	$\{d^2\}$	$\{d, d^3\}$	$\{s, d^2 s\}$	$\{ds, d^3 s\}$
$D^{(1,1)}$	1	1	1	1	1
$D^{(1,2)}$	1	1	1	-1	-1
$D^{(1,3)}$	1	1	-1	1	-1
$D^{(1,4)}$	1	1	-1	-1	1
$D^{(2,1)}$	2	-2	0	0	0

Center of a Group:

$$Z(G) := \{z \in G | zg = gz, \forall g \in G\}$$

Set of all group elements commuting with all elements of G .

Examples:

- $Z(D_n) = \{e\}$ for n odd
- $Z(D_n) = \{e, d^{n/2}\}$ for n even
- $Z(SU(2)) = \{-E, E\}$ $E = 2 \times 2$ unit matrix

Exercise 8: The Group $SU(2)$

Unitary complex 2×2 matrices with unit determinant.

General ansatz:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}$$

Unitarity, $gg^\dagger = 1 = g^\dagger g$, implies:

$$|a|^2 + |b|^2 = 1, |c|^2 + |d|^2 = 1, |a|^2 + |c|^2 = 1, |b|^2 + |d|^2 = 1 \text{ and } a^*c + bd^* = 0, ab^* + cd^* = 0$$

$\det g = 1$ implies: $ad - cb = 1$

Hence:

$$|b|^2 = |c|^2 \text{ and } |a|^2 = |d|^2 \Rightarrow \text{Ansatz: } d = a^* e^{i\alpha} \text{ and } c = -b^* e^{i\beta}$$

$$\Rightarrow ab^* + cd^* = ab^* - b^* a e^{i(\beta-\alpha)} = 0 \Rightarrow \alpha = \beta$$

$$\Rightarrow ad - cb = |a|^2 e^{i\alpha} + |b|^2 e^{i\alpha} = 1 \Rightarrow \alpha = 0$$

$$\Rightarrow d = a^* \text{ and } c = -b^* \text{ with } |a|^2 + |b|^2 = 1$$

Result:

$$g = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad a, b \in \mathbb{C} \quad \text{with} \quad |a|^2 + |b|^2 = 1$$

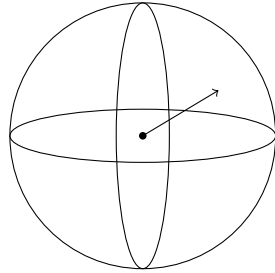
Ansatz:

$$a = \sqrt{1 - \vec{\alpha}^2} + i\alpha_3, \quad b = \alpha_2 + i\alpha_1$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ with $|\vec{\alpha}|^2 \leq 1$.

Comments:

- Group space of $SU(2)$ is unit ball $B^3 := \{\vec{\alpha} \in \mathbb{R}^3 \mid |\vec{\alpha}|^2 \leq 1\}$



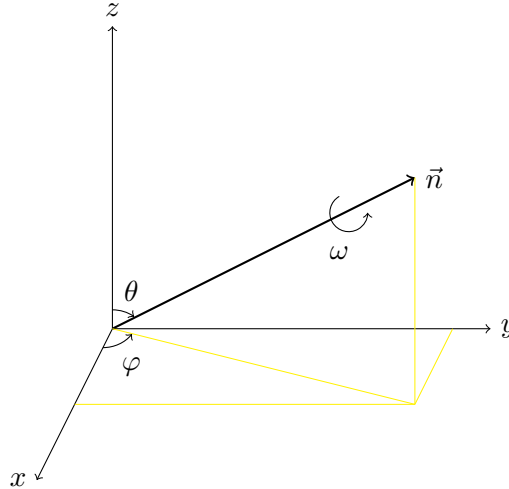
origin $\vec{0}$ represents the unit element

- Group space of $SU(2)$ is the unit sphere $S^3 := \{\vec{x} \in \mathbb{R}^4 \mid |\vec{x}|^2 = 1\}$
Set $\vec{x} = (\text{Re } a, \text{Im } a, \text{Re } b, \text{Im } b)$ with $|\vec{x}|^2 = |a|^2 + |b|^2 = 1$
- $SU(2)$ is a simply connected space = every loop can be contracted to a point
- $SU(2)$ is the universal covering group of $SO(3)$ (see next exercise)

Exercise 9: The Group $SO(3)$

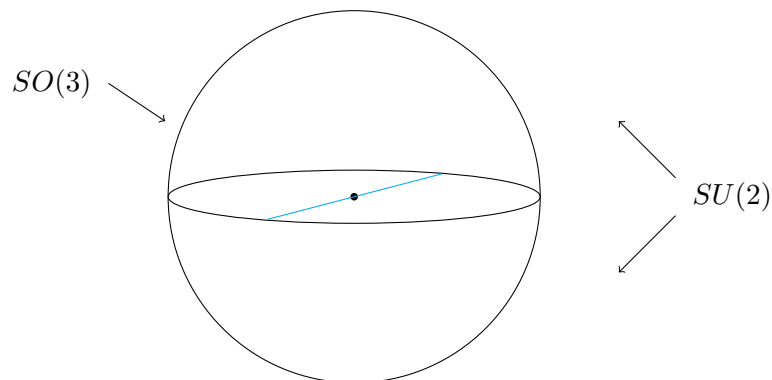
Group of rotations in $\mathbb{R}^3$ characterised by rotation axis $\vec{n} = \vec{n}(\theta, \varphi)$ and rotation angle $\omega \in [0, 2\pi[$:

$$R : \begin{array}{l} \mathbb{R}^3 \rightarrow \mathbb{R}^3 \\ \vec{x} \mapsto R\vec{x} \end{array} \quad \text{with} \quad |R\vec{x}|^2 = |\vec{x}|^2 \quad \text{and} \quad \det R = 1$$



Note: Rotation around $\vec{n}$ by angle ω is equivalent to rotation around $-\vec{n}$ and angle $2\pi - \omega$
 $\Rightarrow$ For $SO(3)$ only upper half of unit sphere formed by $\vec{n}$ and only half of the equator

Group space of $SO(3)$: $\vec{\omega} = \omega\vec{n}$ is half of unit ball B^3 with radius 2π



Group space of $SO(3)$ is doubly connected

Exercise 10: The Connection between $SU(2)$ and $SO(3)$

Consider $\mathcal{M} := \text{span}(\sigma_1, \sigma_2, \sigma_3)$, the vector space of traceless 2×2 matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider mapping

$$\begin{aligned} \mathbb{R}^3 &\rightarrow \mathcal{M} \\ M: \vec{x} &\mapsto M(\vec{x}) := \vec{\sigma} \cdot \vec{x} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \end{aligned}$$

Note $\det M(\vec{x}) = -|\vec{x}|^2$ and the mapping is bijective:

$$\begin{aligned} \vec{x} &= \frac{1}{2} \text{Tr}(M(\vec{x})\vec{\sigma}) \\ &= \frac{\vec{e}_1}{2} \text{Tr}(M(\vec{x})\sigma_1) + \frac{\vec{e}_2}{2} \text{Tr}(M(\vec{x})\sigma_2) + \frac{\vec{e}_3}{2} \text{Tr}(M(\vec{x})\sigma_3) \\ &= \frac{\vec{e}_1}{2} \text{Tr} \begin{pmatrix} x_1 - ix_2 & x_3 \\ -x_3 & x_1 + ix_2 \end{pmatrix} + \frac{\vec{e}_2}{2} \text{Tr} \begin{pmatrix} ix_1 + x_2 & -ix_3 \\ -ix_3 & -ix_1 + x_2 \end{pmatrix} + \frac{\vec{e}_3}{2} \text{Tr} \begin{pmatrix} x_3 & -x_1 + ix_2 \\ x_1 + ix_2 & x_3 \end{pmatrix} \\ &= x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3 \end{aligned}$$

Homomorphism $SU(2) \rightarrow SO(3)$:

$$H: \begin{matrix} SU(2) \rightarrow SO(3) \\ g \mapsto R(g) \end{matrix}, \quad \text{where} \quad M(R(g)\vec{x}) := gM(\vec{x})g^{-1}$$

Note $\det M(R(g))\vec{x} = \det M(\vec{x}) \Rightarrow |R(g)\vec{x}|^2 = |\vec{x}|^2$. That is $R(g)$ is rotation in $\mathbb{R}^3$.

Let $E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $M(R(\pm E)\vec{x}) = (\pm E)M(\vec{x})(\pm E) = M(\vec{x})$.

Kernel of H : $Z_2 := \{E, -E\} \subset SU(2)$

Z_2 is center of $SU(2)$: Center of G is set $Z = \{z \in G | zg = gz\}$, is normal subgroup

$$SO(3) \simeq SU(2)/Z_2$$

Comments:

- Any multiple connected group has a unique universal covering group
see, e.g., B. Wybourne "Classical Groups for Physicists"
- $SO(3)$ is doubly connected $\Rightarrow$ has single and double-valued reps
 $\ell = 0, 1, 2, \dots$ single valued
 $\ell = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ double valued (= single valued for $SU(2)$)
see, e.g., Joshi "Elements of Group Theory for Physicists"

Representations of $SO(3)$ in $\mathbb{R}^3$:

Rodriguez representation:

$$R(\vec{\omega}) := \mathbf{1} \cos \omega + (1 - \cos \omega) \vec{n} \vec{n}^T + \sin \omega \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}, \quad \vec{\omega} = \omega \vec{n}$$

or with

$$N := \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \quad \Rightarrow \quad R(\omega) = \mathbf{1} + \sin \omega N + (1 - \cos \omega) N^2 \quad \text{Proof!}$$

Example: $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow R(\vec{\omega}) = \begin{pmatrix} \cos \omega & -\sin \omega & 0 \\ \sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Euler representation:

$$R(\varphi, \theta, \psi) = R_z(\varphi)R_x(\theta)R_z(\psi)$$

Rotations about x, y and z axis:

$$R_x(\alpha) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}, R_y(\alpha) := \begin{pmatrix} \cos \alpha & 0 & -\sin \alpha \\ \sin \alpha & 0 & \cos \alpha \\ 0 & 0 & 1 \end{pmatrix}, R_z(\alpha) := \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Representations of $SO(3)$ in $L^2(\mathbb{R}^3)$:

$$(D(\vec{\omega})\psi)(\vec{x}) := \psi(R^{-1}(\vec{\omega})\vec{x}), \quad \text{where} \quad D(\vec{\omega}) = \exp\{-i\vec{\omega} \cdot \vec{L}/\hbar\}$$

Angular momentum operator $\vec{L} = \vec{Q} \times \vec{P}$, $[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k$

Group law: $D(\omega\vec{n})D(\omega'\vec{n}) = D((\omega + \omega')\vec{n})$ for fixed axis

Unitarity: $D^{-1}(\vec{\omega}) = D(-\vec{\omega}) = D^\dagger(\vec{\omega})$

Decomposition of Hilbert space: $L^2(\mathbb{R}^3) = L^2(\mathbb{R}^+) \otimes L^2(S^2)$ with $L^2(S^2) = \bigoplus_{\ell=0}^{\infty} \mathcal{D}^\ell$

$\mathcal{D}^\ell = \text{span}\{|\ell m\rangle | m = -\ell, \dots, \ell\}$, $\dim \mathcal{D}^\ell = 2\ell + 1$

Representations of $SO(3)$ in Spin space $\mathbb{C}^{2s+1}$:

$\mathbb{C}^{2s+1} = \text{span}\{|\vec{e}, m\rangle | m = -s, \dots, s\}$, $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

Quantisation axis $\vec{e}$: $\vec{S} \cdot \vec{e} |\vec{e}, m\rangle = m\hbar |\vec{e}, m\rangle$, usually $\vec{e} = \vec{e}_z$

Spin operator: $\vec{S}^2 = s(s+1)\hbar^2$

UIR: $D(\vec{\omega}) = \exp\{-i\vec{\omega} \cdot \vec{S}/\hbar\}$

Rotation of quantisation axis:

$$D(\vec{\omega})(\vec{e} \cdot \vec{S})D^\dagger(\vec{\omega}) = (R(\omega)\vec{e}) \cdot \vec{S}, \quad D(\vec{\omega})|\vec{e}, m\rangle = |R(\vec{\omega})\vec{e}, m\rangle$$

ADDENDUM: $\hbar = 1$

Proof of $D(\vec{\omega}) = \exp\{-i\vec{\omega} \cdot \vec{L}\} = e^{-i\varphi L_3} e^{-i\theta L_2} e^{-i\omega L_3} e^{i\theta L_2} e^{i\varphi L_3}$, $\vec{\omega} = \omega\vec{n}(\theta, \varphi)$

• Note: $L'_3 := e^{-i\theta L_2} L_3 e^{i\theta L_2} \Rightarrow e^{-i\omega L'_3} = e^{-i\theta L_2} e^{-i\omega L_3} e^{i\theta L_2}$ (use $Ae^B A^{-1} = e^{ABA^{-1}}$)

Consider:

$$\frac{\partial L'_3}{\partial \theta} = -ie^{-i\theta L_2} (L_2 L_3 - L_3 L_2) e^{i\theta L_2} = e^{-i\theta L_2} L_1 e^{i\theta L_2} \rightarrow L_1 \quad \text{for } \theta \rightarrow 0$$

$$\frac{\partial^2 L'_3}{\partial \theta^2} = -ie^{-i\theta L_2} (L_2 L_1 - L_1 L_2) e^{i\theta L_2} = -e^{-i\theta L_2} L_3 e^{i\theta L_2} = -L'_3$$

2. order differential eq. with initial conditions $L'_3|_{\theta=0} = L_3$ and $\frac{\partial L'_3}{\partial \theta}|_{\theta=0} = L_1$

Integration yields: $L'_3 = L_3 \cos \theta + L_1 \sin \theta$

Hence

$$\begin{aligned} D(\vec{\omega}) &= e^{-i\varphi L_3} e^{-i\omega L'_3} e^{i\varphi L_3} = \exp\{-i\omega e^{-i\varphi L_3} (L_3 \cos \theta + L_1 \sin \theta) e^{i\varphi L_3}\} \\ &= \exp\{-i\omega (L_3 \cos \theta + L'_1 \sin \theta)\} \\ &\quad \text{with } L'_1 := e^{-i\varphi L_3} L_1 e^{i\varphi L_3} = L_1 \cos \varphi + L_2 \sin \varphi \\ &= \exp\{-i\omega (L_3 \cos \theta + L_1 \sin \theta \cos \varphi + L_2 \sin \theta \sin \varphi)\} = \exp\{-i\omega \vec{n} \cdot \vec{L}\} \end{aligned}$$

Transformation of vector operators

$$\exp\{-i\vec{\omega} \cdot \vec{L}\} \vec{V} \exp\{i\vec{\omega} \cdot \vec{L}\} = R^{-1}(\vec{\omega}) \vec{V}$$

Exercise 10: Rotations in $\mathbb{R}^n$ and the Group $SO(n)$

Good reference:

N.Ja. Vilenkin, *Special Functions and the Theory of Group Representations* (1968)

Let $\vec{x} \in \mathbb{R}^n$ be represented in polar coordinates

$$\begin{aligned} x_1 &= r \sin \vartheta_{n-1} \sin \vartheta_{n-2} \cdots \sin \vartheta_2 \sin \vartheta_1 \\ x_2 &= r \sin \vartheta_{n-1} \sin \vartheta_{n-2} \cdots \sin \vartheta_2 \cos \vartheta_1 \\ x_3 &= r \sin \vartheta_{n-1} \sin \vartheta_{n-2} \cdots \cos \vartheta_2 \\ &\vdots \\ x_{n-1} &= r \sin \vartheta_{n-1} \cos \vartheta_{n-2} \\ x_n &= r \cos \vartheta_{n-1} \end{aligned} \quad \text{where} \quad \begin{aligned} 0 &\leq r < \infty \\ 0 &\leq \vartheta_k \leq \pi, k = 2, 3, \dots, n-1 \\ 0 &\leq \vartheta_1 < 2\pi \end{aligned}$$

Volume element:

$$d^n \vec{x} = r^{n-1} dr d^{n-1} \Omega, \quad d^{n-1} \Omega = \sin^{n-2} \vartheta_{n-1} \sin^{n-3} \vartheta_{n-2} \cdots \sin \vartheta_2 d\vartheta_{n-1} \cdots d\vartheta_1$$

Volume:

$$\int_{S^{n-1}} d^{n-1} \Omega = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

Parametrisation of group elements

Let $g \in SO(n)$ such that $|g\vec{x}| = |\vec{x}|$ and $\det(g) = +1$, i.e. represent rotation matrix in $\mathbb{R}^n$

Let $g_k(\alpha)$ be (clockwise) rotation in plane $(x_k - x_{k+1})$ by angle α

Then each $g \in SO(n)$ can be represented by

$$g = g^{(n-1)} g^{(n-2)} \cdots g^{(1)} \quad \text{where} \quad g^{(k)} := g_1(\theta_1^k) g_2(\theta_2^k) \cdots g_k(\theta_k^k)$$

$$0 \leq \theta_1^k < 2\pi, \quad 0 \leq \theta_j^k \leq \pi, \quad j = 2, 3, \dots, k$$

Obviously $g^{(k)} \in SO(k+1) \subset SO(n)$

Examples:

$n = 2$:

$$g = g_1(\theta_1^1) = \begin{pmatrix} \cos \theta_1^1 & \sin \theta_1^1 \\ -\sin \theta_1^1 & \cos \theta_1^1 \end{pmatrix}$$

$n = 3$:

$$\begin{aligned} g &= g^{(2)} g^{(1)} = g_1(\theta_1^2) g_2(\theta_2^2) g_1(\theta_1^1) \\ &= \begin{pmatrix} \cos \theta_1^2 & \sin \theta_1^2 & 0 \\ -\sin \theta_1^2 & \cos \theta_1^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_2^2 & \sin \theta_2^2 \\ 0 & -\sin \theta_2^2 & \cos \theta_2^2 \end{pmatrix} \begin{pmatrix} \cos \theta_1^1 & \sin \theta_1^1 & 0 \\ -\sin \theta_1^1 & \cos \theta_1^1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Note: Number of group elements $\sum_{k=1}^{n-1} \sum_{j=1}^k 1 = \sum_{k=1}^{n-1} k = \frac{n}{2}(n-1) \Rightarrow \dim SO(n) = n(n-1)/2$

Consider: $h = g^{(n-2)} \dots g^{(1)} \in SO(n-1) \subset SO(n)$ that is

$$h = \left(\begin{array}{c|c} SO(n-1) & 0 \\ \hline 0 & 1 \end{array} \right)$$

Let $\vec{n} = (0, 0, \dots, 0, 1)^T$ be vector to "north pole", then $h\vec{n} = \vec{n}$ is invariant under $SO(n-1)$
 $\Rightarrow$

$$\begin{aligned} g\vec{n} &= g^{(n-1)}\vec{n} = g_1(\theta_1^{n-1}) \dots g_{n-1}(\theta_{n-1}^{n-1})\vec{n} \\ &= g_1(\theta_1^{n-1}) \dots g_{n-2}(\theta_{n-2}^{n-1}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sin \theta_{n-1}^{n-1} \\ \cos \theta_{n-1}^{n-1} \end{pmatrix} \\ &\vdots \\ &= \begin{pmatrix} \sin \theta_{n-1}^{n-1} \sin \theta_{n-2}^{n-1} \dots \sin \theta_1^{n-1} \\ \sin \theta_{n-1}^{n-1} \sin \theta_{n-2}^{n-1} \dots \cos \theta_1^{n-1} \\ \vdots \\ \sin \theta_{n-1}^{n-1} \cos \theta_{n-2}^{n-1} \\ \cos \theta_{n-1}^{n-1} \end{pmatrix} \end{aligned}$$

Obviously $\vec{x}/r = \vec{e}_x = g\vec{n}$ with $\vartheta_k = \theta_k^{(n-1)}$

Invariant Haar measure:

$$\begin{aligned} dg &= \prod_{l=2}^n \frac{\Gamma(l/2)}{2\pi^{l/2}} \prod_{k=1}^{n-1} \left(\prod_{j=1}^k \sin^{j-1} \theta_j^k d\theta_j^k \right) \\ dg &= dh \underbrace{\frac{\Gamma(n/2)}{2\pi^{n/2}} d^{n-1}\Omega}_{d\Gamma} \end{aligned}$$

$d\Gamma$: normalised $SO(n)$ -invariant measure on $S^{n-1} = SO(n)/SO(n-1)$, $\int_{S^{n-1}} d\Gamma = 1$

$$n = 2 : \quad dg = \frac{1}{2\pi} d\theta_1^1$$

$$n = 3 : \quad dg = \frac{1}{8\pi^2} d\theta_1^1 d\theta_1^2 \sin \theta_2^2 d\theta_2^2$$

$$n = 4 : \quad dg = \frac{1}{16\pi^4} d\theta_1^1 d\theta_1^2 \sin \theta_2^2 d\theta_2^2 d\theta_1^3 \sin \theta_2^3 d\theta_2^3 \sin^2 \theta_3^3 d\theta_3^3$$

Representations of class 1

Let $\mathcal{H} = L^2(S^{n-1})$

$$\mathcal{H} = \sum_{\ell=0}^{\infty} \mathcal{H}^{\ell}, \quad \dim \mathcal{H}^{\ell} = (2\ell + n - 2) \frac{(\ell + n - 3)!}{\ell!(n-2)!}, \quad \mathcal{H}^{\ell} = \text{span} \{ |\ell M\rangle \},$$

$$M = (m_1, m_2, \dots, m_{n-2}), \quad \ell, m_1, \dots, m_{n-3} \in \mathbb{N}_0, m_{n-2} \in \mathbb{Z}, \quad \ell \equiv m_0 \geq m_1 \geq \dots \geq m_{n-3} \geq |m_{n-2}|$$

Let $\vec{e}_x \cdot \vec{n} = \cos \theta_{n-1}^{n-1}$ then with $O = (0, 0, \dots, 0)$ and $D_{OO}^{\ell}(g) = \langle \ell O | D^{\ell}(g) | \ell O \rangle$

$$D_{OO}^{\ell}(g) = \frac{\ell! \Gamma(n-2)}{\Gamma(n+\ell-2)} C_{\ell}^{(n-2)/2}(\cos \theta_{n-1}^{n-1}) = D_{OO}^{\ell}(h_1 g h_2), \quad \forall h_1, h_2 \in SO(n-1)$$

form a complete set of zonal spherical functions on S^{n-1} , C_{ℓ}^{ν} Gegenbauer polynomials.

Hyperspherical harmonics:

$$Y_{\ell M}(\vec{e}_x) := \sqrt{d_\ell} D_{MO}^\ell(g) = \sqrt{d_\ell} D_{MO}^\ell(gh), \quad \forall h \in SO(n-1)$$

form orthonormal set on $\mathcal{H}$

$$\int_{S^{n-1}} d\Gamma Y_{\ell M}(\vec{e}_x) Y_{\ell' M'}^*(\vec{e}_x) = \delta_{\ell\ell'} \delta_{MM}$$

Explicit expression

$$Y_{\ell M}(\vec{e}_x) = A_{\ell M} e^{im_{n-2}\vartheta_1} \prod_{k=0}^{n-3} \left[C_{m_k+m_{k+1}}^{m_{k+1}+(n-k-2)/2}(\cos \vartheta_{n-k-1}) \sin \vartheta_{n-k-1} \right]$$

$$A_{\ell M}^2 = \frac{1}{\Gamma(n/2)} \prod_{k=0}^{n-3} \frac{2^{2m_{k+1}+n-k-4} (m_k - m_{k+1})!}{\sqrt{\pi} \Gamma(m_{k+1} + m_k + n - k - 2)} (n - k - 2 + 2m_k) \Gamma^2(m_{k+1} + (n - k - 2)/2)$$

Check for $n = 3$ with $m_0 = \ell$ and $m_1 = m \in \{-\ell, \dots, \ell\}$

Gegenbauer polynomials (or ultra-spherical polynomials):

- Hypergeometric series: $C_n^\alpha(z) = \frac{(2\alpha)_n}{n!} {}_2F_1\left(-n, 2\alpha + n; \alpha + \frac{1}{2}; \frac{1-z}{2}\right)$

Pochhammer symbol: $(a)_n := a(a-1)(a-2)\cdots(a-n+1) = \frac{\Gamma(a)}{\Gamma(a-n)}$

Hypergeom. function: ${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$

- Jacobi polynomials: $C_n^\alpha(z) = \frac{(2\alpha)_n}{(\alpha + \frac{1}{2})_n} P_n^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})}(z)$

- Generalization of Legendre polynomials: $P_n(z) = C_n^{\frac{1}{2}}(z)$ i.e. $\alpha = \frac{1}{2}$

- Expansion in $\mathbb{R}^n$: $\vec{x}, \vec{y} \in \mathbb{R}^n \quad \vec{x} \cdot \vec{y} = xy \cos \theta$

$$\frac{1}{|\vec{x} - \vec{y}|^{n-2}} = \sum_{\ell=0}^{\infty} \frac{x^\ell}{y^{\ell+n-2}} C_{\ell}^{\frac{n-2}{2}}(\cos \theta)$$

*** End of Tutorial 3 ***