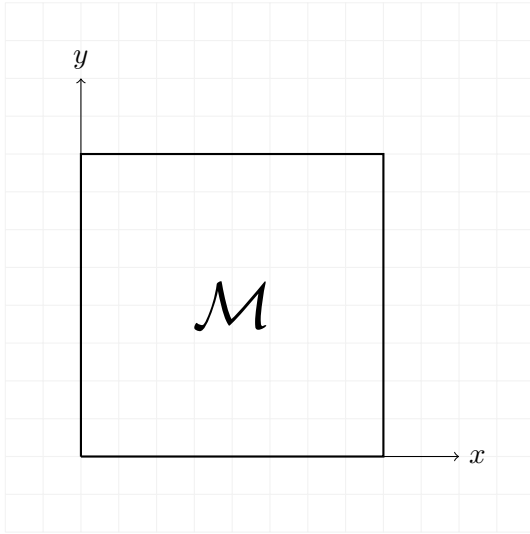
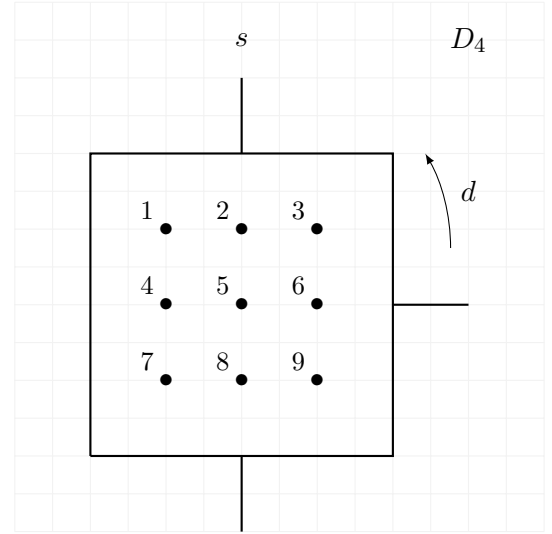


Exercise 4: Eigenmodes of a Membrane

Problem: Find the eigenmodes of a quadratic membrane approximated by 9 lattice sites



$\xi(x, y)$ obeys
 $-\Delta \xi(x, y) = \omega^2 \xi(x, y)$
 with $\xi(\partial \mathcal{M}) = 0$



$\vec{\xi} = (\xi_1, \xi_2, \dots, \xi_9) \in \mathbb{R}^9 =: V$
 EV problem $M\vec{\xi} = \lambda\vec{\xi}$
 M suitable 9×9 matrix

1) Choose proper Symmetry Group

$D_4 = \{e, d, d^2, d^3, s, sd, sd^2, sd^3\}$, $d^4 = e = s^2$, $sd = d^{-1}s = d^3s$

• Basis in V :

$$\vec{e}_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} =: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \dots \quad \vec{e}_9 := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} =: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

• Representation of generators in V :

$$s \mapsto D^{(9)}(s) = \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & & & \\ 0 & 1 & 0 & & 0 & \\ 1 & 0 & 0 & & 0 & \\ \hline & 0 & & 0 & 0 & 1 \\ & & & 0 & 1 & 0 \\ & & & 1 & 0 & 0 \\ \hline & 0 & & & & \\ & & & & & 0 & 0 & 1 \\ & & & & & 0 & 1 & 0 \\ & & & & & 1 & 0 & 0 \end{array} \right) \quad \text{as} \quad \begin{array}{l} 3 \rightarrow 1 \\ 2 \rightarrow 2 \\ 1 \rightarrow 3 \\ 6 \rightarrow 4 \\ 5 \rightarrow 5 \\ 4 \rightarrow 6 \\ 9 \rightarrow 7 \\ 8 \rightarrow 8 \\ 7 \rightarrow 9 \end{array}$$

$$d \mapsto D^{(9)}(d) = \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right) \quad \text{as} \quad \begin{array}{l} 3 \rightarrow 1 \\ 6 \rightarrow 2 \\ 9 \rightarrow 3 \\ 2 \rightarrow 4 \\ 5 \rightarrow 5 \\ 8 \rightarrow 6 \\ 1 \rightarrow 7 \\ 4 \rightarrow 8 \\ 7 \rightarrow 9 \end{array}$$

Let $\vec{\xi}_g := D^{(9)}(g)\vec{\xi}$ then $\vec{\xi}_g$ is also eigenvector of M to same eigenvalue λ :

$$MD^{(9)}(g) = D^{(9)}(g)M \quad \text{for all } g \in D_4$$

M has D_4 symmetry

2) UIR of D_4

Without proof: D_4 has four 1-dim. and one 2-dim. UIR (see Stiefel & Fässler)
 $\text{ord } D_4 = 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$ Burnside's theorem (later)

1-dim. UIR: Notation of Stiefel & Fässler

$$\begin{array}{ll} D^{(1,1)} : \begin{array}{l} d \mapsto 1 \\ s \mapsto 1 \end{array} & \text{trivial reps.} \\ D^{(1,3)} : \begin{array}{l} d \mapsto -1 \\ s \mapsto 1 \end{array} & \end{array} \quad \begin{array}{ll} D^{(1,2)} : \begin{array}{l} d \mapsto 1 \\ s \mapsto -1 \end{array} \\ D^{(1,4)} : \begin{array}{l} d \mapsto -1 \\ s \mapsto -1 \end{array} \end{array}$$

2-dim. UIR:

$$\begin{array}{l} D^{(2,1)} : \\ \begin{array}{l} d \mapsto D^{(2,1)}(d) = \begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ s \mapsto D^{(2,1)}(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \end{array}$$

3) Find Invariant Subspaces

3×1 -dim. inv. subspaces belonging to $D^{(1,1)}$

$$\vec{x}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

obviously $D^{(9)}(g)\vec{x}_i = \vec{x}_i$ for all $g \in D_4$

1×1 -dim. inv. subspace belonging to $D^{(1,3)}$

$$\vec{y} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \left. \begin{array}{l} D^{(9)}(d)\vec{y} = -\vec{y} \\ D^{(9)}(s)\vec{y} = \vec{y} \end{array} \right\} \quad 1\text{-dim. reps } D^{(1,3)}(g)$$

1×1 -dim. inv. subspace belonging to $D^{(1,4)}$

$$\vec{z} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \left. \begin{array}{l} D^{(9)}(d)\vec{z} = -\vec{z} \\ D^{(9)}(s)\vec{z} = -\vec{z} \end{array} \right\} \quad 1\text{-dim. reps } D^{(1,4)}(g)$$

2×2 -dim. inv. subspaces belonging to $D^{(1,2)}$

$$\vec{u}_1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\left. \begin{array}{l} D^{(9)}(d)\vec{u}_1 = \vec{u}_2 \\ D^{(9)}(d)\vec{u}_2 = -\vec{u}_1 \end{array} \right\} D^{(9)}(d)|_{\vec{u}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \left. \begin{array}{l} D^{(9)}(s)\vec{u}_1 = -\vec{u}_1 \\ D^{(9)}(s)\vec{u}_2 = \vec{u}_2 \end{array} \right\} D^{(9)}(s)|_{\vec{u}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

$$\left. \begin{array}{l} D^{(9)}(d)\vec{v}_1 = \vec{v}_2 \\ D^{(9)}(d)\vec{v}_2 = -\vec{v}_1 \end{array} \right\} D^{(9)}(d)|_{\vec{v}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \left. \begin{array}{l} D^{(9)}(s)\vec{v}_1 = -\vec{v}_1 \\ D^{(9)}(s)\vec{v}_2 = \vec{v}_2 \end{array} \right\} D^{(9)}(s)|_{\vec{v}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Conclusion:

$$D^{(9)}(g) = 3 \cdot D^{(1,1)}(g) + 1 \cdot D^{(1,3)}(g) + 1 \cdot D^{(1,4)}(g) + 2 \cdot D^{(2,1)}(g)$$

$D^{(9)}$ becomes block-diagonal under change of basis

$$\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_9\} \Rightarrow \{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{y}, \vec{z}, \vec{u}_1, \vec{u}_2, \vec{v}_1, \vec{v}_2\}$$

4) Constructing Operator M

Remember: $\frac{d\xi(x, y)}{dx} \Rightarrow \frac{\xi_i - \xi_{i-1}}{a} =: \frac{\Delta\xi_i}{a}$ and $\frac{d^2\xi(x, y)}{dx^2} \Rightarrow \frac{\Delta\xi_{i+1} - \Delta\xi_{i-1}}{a^2} = \frac{\xi_{i+1} - 2\xi_i + \xi_{i-1}}{a^2}$

Hence, with $a = 1$ we replace $-\frac{\partial^2\xi}{\partial x^2} \Rightarrow M_{xx}\xi_i = -\xi_{i-1} + 2\xi_i - \xi_{i+1}$, where $\xi_{i\pm 1}$ is the excitation to left/right of point i . If there is no left/right this is set to zero (boundary condition). Similarly, we replace $-\frac{\partial^2\xi}{\partial y^2} \Rightarrow M_{yy}\xi_i = -\xi_{i-3} + 2\xi_i - \xi_{i+3}$, where $\xi_{i\pm 3}$ is the excitation to above/below of point i . If there is no above/below this is set to zero (boundary condition).

The matrix $M = M_{xx} + M_{yy}$ is therefore represented by

$M\vec{e}_1 = 4\vec{e}_1 - \vec{e}_2 - \vec{e}_4$	only right and below neighbor
$M\vec{e}_2 = -\vec{e}_1 + 4\vec{e}_2 - \vec{e}_3 - \vec{e}_5$	only left/right and below
$M\vec{e}_3 = -\vec{e}_2 + 4\vec{e}_3 - \vec{e}_6$	only left and below
$M\vec{e}_4 = -\vec{e}_1 + 4\vec{e}_4 - \vec{e}_5 - \vec{e}_7$	only right and above/below
$M\vec{e}_5 = -\vec{e}_2 - \vec{e}_4 + 4\vec{e}_5 - \vec{e}_6 - \vec{e}_8$	all four neighbours
$M\vec{e}_6 = -\vec{e}_3 - \vec{e}_5 + 4\vec{e}_6 - \vec{e}_9$	only left and above/below
$M\vec{e}_7 = -\vec{e}_4 + 4\vec{e}_7 - \vec{e}_8$	only right and above
$M\vec{e}_8 = -\vec{e}_5 - \vec{e}_7 + 4\vec{e}_8 - \vec{e}_9$	only left and above/below
$M\vec{e}_9 = -\vec{e}_6 - \vec{e}_8 + 4\vec{e}_9$	only left and above

Hence M in old basis is given by

$$M = \left(\begin{array}{ccc|ccc|ccc} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{array} \right)$$

has D_4 -symmetry as $MD^{(9)}(g) = D^{(9)}(g)M$ for all $g \in D_4$ (check for $g = \{d, s\}$!!!)

Construct now M in new basis

$$M\vec{x}_1 = M\vec{e}_5 = -\vec{e}_2 - \vec{e}_4 + 4\vec{e}_5 - \vec{e}_6 - \vec{e}_8 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix} = 4\vec{x}_1 - \vec{x}_2$$

$$M\vec{x}_2 = M(\vec{e}_2 + \vec{e}_4 + \vec{e}_6 + \vec{e}_8) = \begin{pmatrix} -2 & 4 & -2 \\ 4 & -4 & 4 \\ -2 & 4 & -2 \end{pmatrix} = -4\vec{x}_1 + 4\vec{x}_2 - 2\vec{x}_3$$

$$M\vec{x}_3 = M(\vec{e}_1 + \vec{e}_3 + \vec{e}_7 + \vec{e}_9) = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 0 & -2 \\ 4 & -2 & 4 \end{pmatrix} = -2\vec{x}_2 + 4\vec{x}_3$$

$\Sigma^{(1,1)} = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is the invariant subspace belonging to the UIR $D^{(1,1)}$

$$M\vec{y} = M(\vec{e}_2 - \vec{e}_4 - \vec{e}_6 + \vec{e}_8) = \begin{pmatrix} 0 & 4 & 0 \\ -4 & 0 & -4 \\ 0 & 4 & 0 \end{pmatrix} = 4\vec{y} \quad \Sigma^{(1,3)}$$

$$M\vec{z} = M(-\vec{e}_1 + \vec{e}_3 + \vec{e}_7 - \vec{e}_9) = \begin{pmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{pmatrix} = 4\vec{z} \quad \Sigma^{(1,4)}$$

$$M\vec{u}_1 = M(-\vec{e}_4 + \vec{e}_6) = \begin{pmatrix} 1 & 0 & -1 \\ -4 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix} = 4\vec{u}_1 - \vec{v}_1 \quad M\vec{u}_2 = M(\vec{e}_2 - \vec{e}_8) = \begin{pmatrix} -1 & 4 & -1 \\ 0 & 0 & 0 \\ 1 & -4 & 1 \end{pmatrix} = 4\vec{u}_2 - \vec{v}_2$$

$$M\vec{v}_1 = M(-\vec{e}_1 + \vec{e}_3 - \vec{e}_7 + \vec{e}_9) = \begin{pmatrix} -4 & 0 & 4 \\ 2 & 0 & -2 \\ -4 & 0 & 4 \end{pmatrix} = -2\vec{u}_1 + 4\vec{v}_1$$

$$M\vec{v}_2 = M(\vec{e}_1 + \vec{e}_3 - \vec{e}_7 - \vec{e}_9) = \begin{pmatrix} 4 & -2 & 4 \\ 0 & 0 & 0 \\ -4 & 2 & -4 \end{pmatrix} = -2\vec{u}_2 + 4\vec{v}_2$$

$\Sigma^{(2,1)} = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{v}_1, \vec{v}_2\}$ is the invariant subspace belonging to the UIR $D^{(2,1)}$

In new basis

$$M = \left(\begin{array}{ccc|c|c|c|c} 4 & -4 & 0 & & & & \\ -1 & 4 & -2 & & & & \\ 0 & -2 & 4 & & & & \\ \hline & & & 4 & & & \\ \hline & & & & 4 & & \\ \hline & & & & & 4 & 0 & 0 & -2 & 0 \\ & & & & & & 0 & 4 & 0 & -2 \\ & & & & & & -1 & 0 & 4 & 0 \\ & & & & & & 0 & -1 & 0 & 4 \end{array} \right) \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \hline y \\ \hline z \\ \hline u_1 \\ u_2 \\ v_1 \\ v_2 \end{array}$$

Note that submatrix can be written as Kronecker product

$$\left(\begin{array}{cc|cc} 4 & 0 & -2 & 0 \\ 0 & 4 & 0 & -2 \\ \hline -1 & 0 & 4 & 0 \\ 0 & -1 & 0 & 4 \end{array} \right) = \begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

change basis $\{u_1, u_2, v_1, v_2\} \Rightarrow \{u_1, v_1, u_2, v_2\}$ results in

$$M = \left(\begin{array}{ccc|c|c|c|c} 4 & -4 & 0 & & & & \\ -1 & 4 & -2 & & & & \\ 0 & -2 & 4 & & & & \\ \hline & & & 4 & & & \\ \hline & & & & 4 & & \\ \hline & & & & & 4 & -2 \\ & & & & & -1 & 4 \\ & & & & & & 4 & -2 \\ & & & & & & -1 & 4 \end{array} \right) \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \hline y \\ \hline z \\ \hline u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

Eigenvalues and eigenvectors are now straightforward, at most 3×3 matrix problem, and are shown graphically on next page

$\Sigma^{(1,1)}$:

$$\begin{aligned} \begin{pmatrix} 4 & -4 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ \sqrt{2} \\ 1 \end{pmatrix} &= (4 - 2\sqrt{2}) \begin{pmatrix} 2 \\ \sqrt{2} \\ 1 \end{pmatrix} \\ \begin{pmatrix} 4 & -4 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} &= 4 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 4 & -4 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -\sqrt{2} \\ 1 \end{pmatrix} &= (4 + 2\sqrt{2}) \begin{pmatrix} 2 \\ -\sqrt{2} \\ 1 \end{pmatrix} \end{aligned}$$

$\Sigma^{(2,1)}$:

$$\begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} = (4 - \sqrt{2}) \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} = (4 + \sqrt{2}) \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

Eigenvalue	Eigenvector	Illustration
$4 - 2\sqrt{2}$	$2\vec{x}_1 + \sqrt{2}\vec{x}_2 + \vec{x}_3$	$\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$
$4 - \sqrt{2}$	$\sqrt{2}\vec{u}_1 + \vec{v}_1$	$\begin{pmatrix} -1 & 0 & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ -1 & 0 & 1 \end{pmatrix}$
$4 - \sqrt{2}$	$\sqrt{2}\vec{u}_2 + \vec{v}_2$	$\begin{pmatrix} 1 & \sqrt{2} & 1 \\ 0 & 0 & 0 \\ -1 & -\sqrt{2} & -1 \end{pmatrix}$
4	$-2\vec{x}_1 + \vec{x}_3$	$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix}$
4	$\vec{y}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$
4	$\vec{z}$	$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$
$4 + \sqrt{2}$	$\vec{v}_1 - \sqrt{2}\vec{u}_1$	$\begin{pmatrix} 1 & 0 & -1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & 0 & -1 \end{pmatrix}$
$4 + \sqrt{2}$	$\vec{v}_2 - \sqrt{2}\vec{u}_2$	$\begin{pmatrix} -1 & \sqrt{2} & -1 \\ 0 & 0 & 0 \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$
$4 + 2\sqrt{2}$	$2\vec{x}_1 - \sqrt{2}\vec{x}_2 + \vec{x}_3$	$\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$

Compare with exact result

$$\begin{aligned}\lambda_{mn} &= \frac{\pi^2}{a^2}(n^2 + m^2) \quad m, n = 1, 2, 3, \dots \\ \xi_{mn}(x, y) &= N_{mn} \sin\left(m\frac{\pi x}{a}\right) \sin\left(n\frac{\pi y}{a}\right)\end{aligned}$$

Lowest mode

$$\begin{aligned}2N_{11}\xi_{11}\left(\frac{a}{4}, \frac{3a}{4}\right) &= 1 & 2N_{11}\xi_{11}\left(\frac{a}{2}, \frac{3a}{4}\right) &= \sqrt{2} & 2N_{11}\xi_{11}\left(\frac{3a}{4}, \frac{3a}{4}\right) &= 1 \\ 2N_{11}\xi_{11}\left(\frac{a}{4}, \frac{a}{2}\right) &= \sqrt{2} & 2N_{11}\xi_{11}\left(\frac{a}{2}, \frac{a}{2}\right) &= 2 & 2N_{11}\xi_{11}\left(\frac{3a}{4}, \frac{a}{2}\right) &= \sqrt{2} \\ 2N_{11}\xi_{11}\left(\frac{a}{4}, \frac{a}{4}\right) &= 1 & 2N_{11}\xi_{11}\left(\frac{a}{2}, \frac{a}{4}\right) &= \sqrt{2} & 2N_{11}\xi_{11}\left(\frac{3a}{4}, \frac{a}{4}\right) &= 1\end{aligned}$$

$$\Rightarrow \xi_{11} \sim \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix} = 2\vec{x}_1 + \sqrt{2}\vec{x}_2 + \vec{x}_3$$

Quality of eigenvalues:

exact (degen.)		approx. (degen.)	
$\frac{\lambda_{12}}{\lambda_{11}} = \frac{5}{2} = 2, 5$	(2)	$\frac{4 - \sqrt{2}}{4 - 2\sqrt{2}} = 2, 2$	(2)
$\frac{\lambda_{22}}{\lambda_{11}} = \frac{8}{2} = 4$	(1)	$\frac{4}{4 - 2\sqrt{2}} = 3, 4$	(3)
$\frac{\lambda_{13}}{\lambda_{11}} = \frac{10}{2} = 5$	(2)	$\frac{4 + \sqrt{2}}{4 - 2\sqrt{2}} = 4, 6$	(2)

Exercise 5: Maschke's Theorem

Theorem: Each representation of a finite group is equivalent to a unitary representation with respect to scalar product

$$(x, y) := \sum_{i=1}^d x_i^* y_i \quad x, y \in V, \quad d = \dim V$$

Proof: Consider a non-unitary reps $D(g)$, that is, with $x^{(i)} := G(g_i)x$, $g_i \in G$ follows

$$(D(g_i)x, D(g_i)y) = (x^{(i)}, y^{(i)}) \neq (x, y)$$

Consider

$$\begin{aligned} \sum_{i=1}^{\text{ord } G} (x^{(i)}, y^{(i)}) &= \sum_{g \in G} (D(g)x, D(g)y) \\ &= \sum_{g \in G} \underbrace{(D^\dagger(g)D(g))}_{\neq 1} x, y =: (Hx, y) \end{aligned} \quad (1)$$

with $H := \sum_{g \in G} D^\dagger(g)D(g) = H^\dagger \geq 0$

$\Rightarrow$ there exists a unitary U which diagonalize H (eigenvalue problem), that is

$$d := U^\dagger H U = \sum_g U^\dagger D^\dagger(g) U U^\dagger D(g) U =: \sum_g \tilde{D}^\dagger(g) \tilde{D}(g) \geq 0$$

As $d = d^\dagger \geq 0$ operator $d^{1/2}$ is well defined $\Rightarrow$

$$(Hx, y) = (U d U^\dagger x, y) = (d U^\dagger x, U^\dagger y) = (d^{1/2} U^\dagger x, d^{1/2} U^\dagger y) = (Lx, Ly) \quad (2)$$

with $L := d^{1/2} U^\dagger$. Hence, $D'(g) := L D(g) L^{-1}$ is a unitary reps in the new basis $x' := Lx$ and $y' := Ly$.

Proof:

$$\begin{aligned} (D'(g)x', D'(g)y') &= (L D(g) L^{-1} x', L D(g) L^{-1} y') \\ &= (L D(g)x, L D(g)y) \stackrel{(2)}{=} (H D(g)x, D(g)y) \\ &\stackrel{(1)}{=} \sum_{\tilde{g}} (D(\tilde{g}) D(g)x, D(\tilde{g}) D(g)y) \\ &= \sum_{\tilde{g}} (D(\tilde{g}g)x, D(\tilde{g}g)y) = \sum_{\tilde{g}} (D(\tilde{g})x, D(\tilde{g})y) \\ &\stackrel{(1)}{=} (Hx, y) \stackrel{(2)}{=} (Lx, Ly) = (x', y') \quad \text{for all } g \in G \end{aligned}$$

Corollary: Each representation D of a finite group is unitary with respect to the scalar product

$$\langle x, y \rangle := \frac{1}{\text{ord } G} \sum_{g \in G} (D(g)x, D(g)y)$$

Proof: For all $g_0 \in G$

$$\begin{aligned} \langle D(g_0)x, D(g_0)y \rangle &= \frac{1}{\text{ord } G} \sum_{g \in G} (D(g)D(g_0)x, D(g)D(g_0)y) \\ &= \frac{1}{\text{ord } G} \sum_{g \in G} (D(gg_0)x, D(gg_0)y) = \frac{1}{\text{ord } G} \sum_{g \in G} (D(g)x, D(g)y) = \langle x, y \rangle \end{aligned}$$

Comment: Extension to compact and even uni-modular groups obvious by replacing

$$\frac{1}{\text{ord } G} \sum_{g \in G} (\cdot) \quad \text{by} \quad \int_G dg (\cdot)$$

Exercise 6: The UIRs of D_n

From homework (problem 6) we know that

$$\begin{aligned} D_n & \text{ has } 4 + \left(\frac{n}{2} - 1\right) \text{ classes for } n \text{ even} \\ D_n & \text{ has } 2 + \left(\frac{n-1}{2}\right) \text{ classes for } n \text{ odd} \end{aligned}$$

From Burnside we know

$$\text{ord } D_n = 2n = \sum_j d_j^2 = \begin{cases} 4 \cdot 1^2 + \left(\frac{n}{2} - 1\right) \cdot 2^2 & n \text{ even} \\ 2 \cdot 1^2 + \left(\frac{n-1}{2}\right) \cdot 2^2 & n \text{ odd} \end{cases}$$

So we expect for

n even: 4 1-dim. and $(n/2 - 1)$ 2-dim. UIR

n odd: 2 1-dim. and $(n - 1)/2$ 2-dim. UIR

From exercise 4 we potentially know four 1-dim. UIR

1-dimensional UIRs

n even:

$$\begin{aligned} D^{(1,1)}(d) &= 1, & D^{(1,1)}(s) &= 1 \\ D^{(1,2)}(d) &= 1, & D^{(1,2)}(s) &= -1 \\ D^{(1,3)}(d) &= -1, & D^{(1,3)}(s) &= 1 \\ D^{(1,4)}(d) &= -1, & D^{(1,4)}(s) &= -1 \end{aligned}$$

n odd:

$$\begin{aligned} D^{(1,1)}(d) &= 1, & D^{(1,1)}(s) &= 1 \\ D^{(1,2)}(d) &= 1, & D^{(1,2)}(s) &= -1 \end{aligned}$$

Obviously the last two of case n even cannot be reps for n odd as $d^n = e$

2-dimensional UIRs

n even: Ansatz

$$\tilde{D}^{(2,r)}(d) = \begin{pmatrix} \cos \alpha_r & -\sin \alpha_r \\ \sin \alpha_r & \cos \alpha_r \end{pmatrix} \quad \text{with} \quad \alpha_r = 2\pi \frac{r}{n} \quad \text{as} \quad d^n = e$$

Note that rotation by φ and $2\pi - \varphi$ are equivalent due to $s \Rightarrow \alpha_r < \pi$
 $\Rightarrow r = 1, 2, 3, \dots, n/2 - 1$

Recall also

$$\tilde{D}^{(2,r)}(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Equivalent reps:

$$D^{(2,r)}(d) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tilde{D}^{(2,r)}(d) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} e^{2\pi i r/n} & 0 \\ 0 & e^{-2\pi i r/n} \end{pmatrix}$$

$$D^{(2,r)}(s) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tilde{D}^{(2,r)}(s) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

n odd: Same as above but now $r = 1, 2, 3, \dots, (n - 1)/2$ as $\alpha_r = 2\pi r/n < \pi$.

Comment: In literature often

$$D^{(2,r)}(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

*** End of Tutorial 2 ***