Group Theory for Physicists

Tutorial

Georg Junker

Friedrich-Alexander-Universität Erlangen-Nürnberg

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Exercise 1: The Permutation Group S_n

Recall from lecture

$$\operatorname{ord} S_n = n!\,, \qquad P = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi_1 & \pi_2 & \pi_3 & \cdots & \pi_n \end{pmatrix}, \qquad \pi_i \in \{1,2,3.\dots,n\}\,, \qquad \pi_i \neq \pi_j \text{ for } i \neq j$$

a) Cayley's Theorem

Theorem: Every group of order $n < \infty$ is isomorphic to a subgroup of S_n

Proof: Let $G := \{g_1, g_2, \dots, g_n\}$
$\Rightarrow$ left multiplication with a fixed $g \in G$ corresponds to a row in Cayley's table for G.
$\Rightarrow$ $G = \{gg_1, gg_2, \dots, gg_n\} =: \{g_{\pi_1}, g_{\pi_2}, \dots, g_{\pi_n}\}$ with $\pi_i \neq \pi_j$ for $i \neq j$

$$\Rightarrow \quad \exists \text{ isomorphism} \quad P: \begin{array}{l} G \to H \subset S_n \\ g \mapsto P(g) := \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi_1 & \pi_2 & \pi_3 & \cdots & \pi_n \end{pmatrix} \end{array}$$

Obviously for $g_1 \neq g_2 \Rightarrow P(g_1) \neq P(g_2)$ as they correspond to different rows in group table.
In addition, $P(g_1)P(g_2) = P(g_1 g_2)$ as here $g_{\pi_i} = g_1(g_2 g_i) = (g_1 g_2) g_i$
$\Rightarrow$ $H \simeq G$ and $\operatorname{ord} H = n$ $\Rightarrow$ H is subgroup of S_n

Remarks:

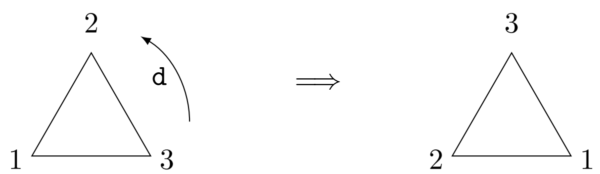
- $C_n \subset D_n \subset S_n$ for $n \geq 3$
  C_n and D_n are symmetry groups of regular n-polygon $\Rightarrow$ permutations of edges
- As $\operatorname{ord} D_3 = 6 = \operatorname{ord} S_3 \Rightarrow D_3 \simeq S_3$

b) The Group S_3

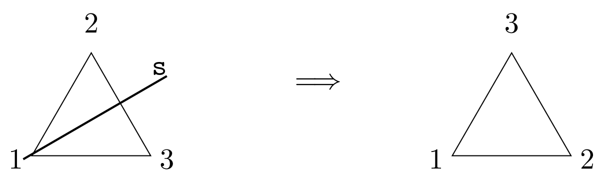
Let

$$e := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \qquad a := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$c := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad d := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \qquad f := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$



rotation $\mathtt{d} \in D_3 \Leftrightarrow a \in S_3$



reflexion $\mathtt{s} \in D_3 \Leftrightarrow c \in S_3$

In general

D_3	e	d	d^2	s	sd	sd^2
S_3	e	a	b	c	f	d

Show for the elements of S_3: $b^2 = a$, $cb = f$ and $ca = d$

Conjugacy Classes:
Remember a class is defined by one element $g \in G$ via

$$\{g_1 g g_1^{-1}, g_2 g g_2^{-1}, \dots, g_n g g_n^{-1}\}$$

- $\{e\} \simeq \{e\}$ obvious
- $\{d, d^2\} \simeq \{a, b\}$ follows from $sd = d^2 s = d^{-1} s$
- $\{s, sd, sd^2\} \simeq \{c, d, f\}$ follows also from $sd = d^2 s = d^{-1} s$

c) Decomposition into Cycles and Transpositions

Cycles: More efficient notation for an element of S_n
Examples:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 4 & 8 & 5 & 7 & 2 & 3 \end{pmatrix} = \left(\begin{array}{cccc|ccc|c} 1 & 6 & 7 & 2 & 3 & 4 & 8 & 5 \\ 6 & 7 & 2 & 1 & 4 & 8 & 3 & 5 \end{array}\right) =: (1672)(348)(5)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 4 & 1 & 8 & 9 & 6 & 7 & 2 \end{pmatrix} = (134)(258769)$$

Cycles have no common elements $\Rightarrow$ commute
Cycles with only one element are trivial and may be ommited

Transposition: Cycles with two elements $[n_1 n_2] := (n_1 n_2)$
Each cycle with $k > 1$ elements may be decomposed into an *ordered* product of $k-1$ transpositions.

$$(n_1 n_2 \cdots n_k) = [n_1 n_k][n_1 n_{k-1}] \cdots [n_1 n_3][n_1 n_2]$$

Proof by induction:
$k = 2$ obvious (see definition)

$$\begin{aligned}
(n_1 n_2 \cdots n_k n_{k+1}) &= \left(\begin{array}{cccccc|c} n_1 & n_2 & \cdots & n_{k-1} & n_k & n_{k+1} & \cdots \\ n_2 & n_3 & \cdots & n_k & n_{k+1} & n_1 & \cdots \end{array}\right) \\
&= \begin{pmatrix} n_1 & n_2 & n_3 & \cdots & n_k & n_{k+1} \\ n_{k+1} & n_2 & n_3 & \cdots & n_k & n_1 \end{pmatrix} \begin{pmatrix} n_1 & n_2 & n_3 & \cdots & n_k & n_{k+1} \\ n_2 & n_3 & n_4 & \cdots & n_1 & n_{k+1} \end{pmatrix} \\
&= (n_1 n_{k+1})(n_1 n_2 \cdots n_k) \\
&= [n_1 n_{k+1}][n_1 n_k] \cdots [n_1 n_3][n_1 n_2]
\end{aligned}$$

Conclusion: Each permutation may be decomposed into a product of transpositions

even permutations $:\Leftrightarrow$ even number of transpositions
odd permutations $:\Leftrightarrow$ odd number of transpositions

Show group homomorphism: $S_n \to C_2$

Example S_3:

S_3	Cycle	transpositions	even/odd
e	( )	[ ]	even
a	(123)	[13][12]	even
b	(132)	[12][13]	even
c	(23)	[23]	odd
d	(13)	[13]	odd
f	(12)	[12]	odd

d) The Alternating Group A_n

The set of even permutations forms a normal subgroup of S_n.
This subgroup is called alternating group A_n, ord $A_n = \frac{1}{2} n!$

e) Generators of S_n

Obviously the transpositions generate the permutations.
Let

$$P_i := [i, i+1] = (i, i+1) = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots \\ 1 & 2 & \cdots & i+1 & i & \cdots \end{pmatrix}, \qquad i = 1, 2, 3, \ldots, n-1$$

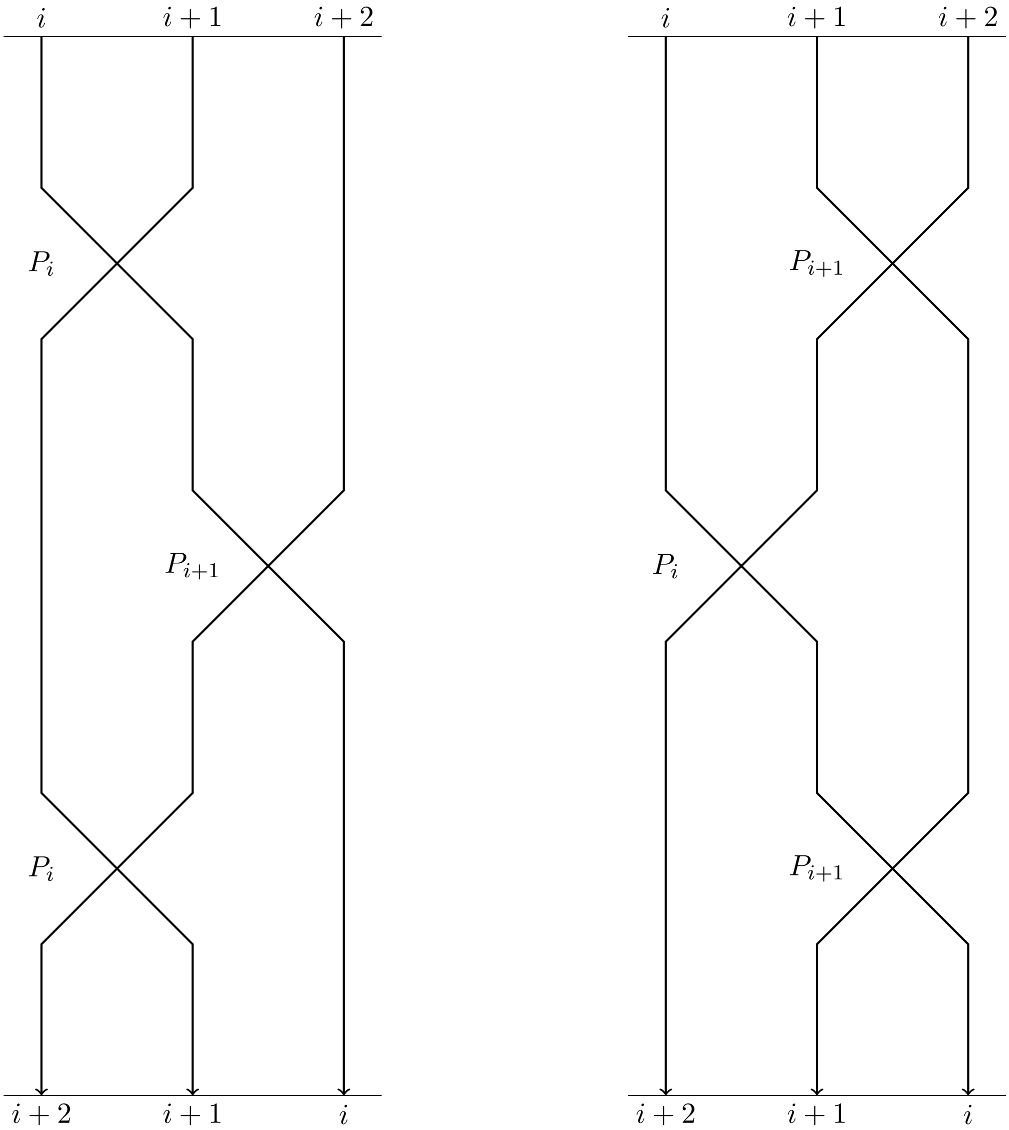
Then

$$P_i = P_i^{-1}, \qquad P_i^2 = e, \qquad P_i P_j = P_j P_i \quad \text{for} \quad |i-j| > 1$$

and

$$\boxed{P_i P_{i+1} P_i = P_{i+1} P_i P_{i+1}}$$

Graphical proof:



Exercise 2: The Braid Group B_n

Generators: $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}\} \in B_n$
with

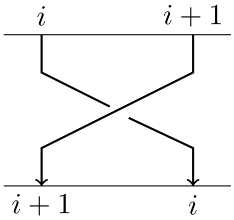
$$\varepsilon_i \varepsilon_{i+1} \varepsilon_i = \varepsilon_{i+1} \varepsilon_i \varepsilon_{i+1}\,, \qquad \varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i \quad \text{for} \quad |i-j| > 1$$

but

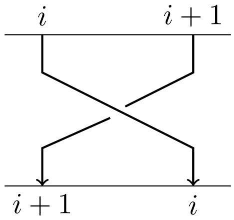
$$\varepsilon_i \neq \varepsilon_i^{-1}\,, \qquad \varepsilon_i^2 \neq e$$

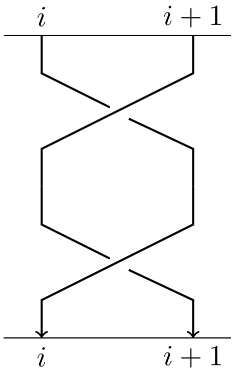
Interpretation: Set of all possible braids made out of n strips.
ε_i = exchange string i and $i+1$ counterclockwise

Graphical representation:

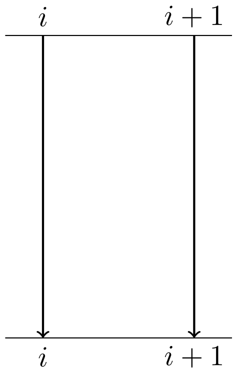


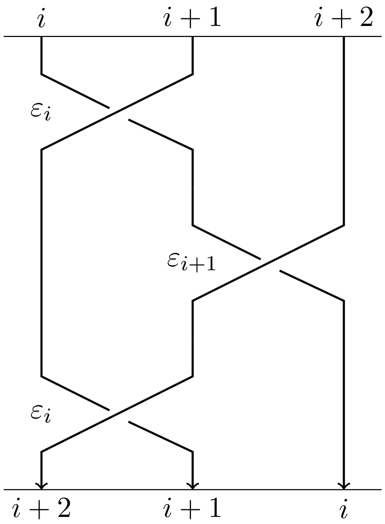
$\varepsilon_i \neq \varepsilon_i^{-1}$



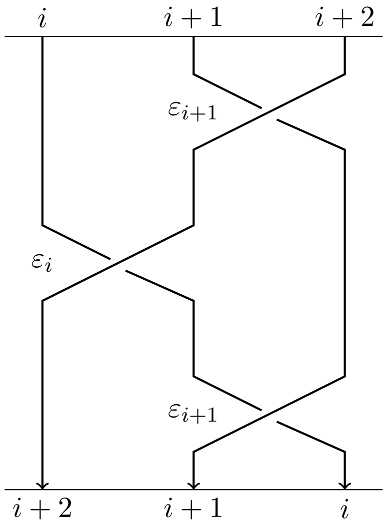


$\varepsilon_i^2 \neq e$





$\varepsilon_i \varepsilon_{i+1} \varepsilon_i = \varepsilon_{i+1} \varepsilon_i \varepsilon_{i+1}$



Remarks:

- If we assume that braids can penetrate each other $\Rightarrow \varepsilon_i^2 = e$ and $\varepsilon_i^{-1} = \varepsilon_i \Rightarrow S_n$
- However $S_n \nsubseteq B_n$, i.e. S_n is NOT a subgroup of B_n
- $B_2 \simeq \mathbb{Z}$ has only one generator ε_1
  all group elements are powers of ε_1, ε_1^m with $m \in \mathbb{Z}$, $\varepsilon_1^0 =: e$
  m is the winding number and uniquely characterises an element of B_2.
  $\mathbb{Z} \simeq \pi_1(S^1)$ *fundamental group* of the unit circle

Exercise 3: Direct Product of Groups

Defintion: The *direct product* $G_1 \otimes G_2$ of two groups G_1 and G_2 forms a group

$$G_1 \otimes G_2 := \{(g_1, g_2)|g_1 \in G_1, g_2 \in G_2\}$$

if all elements of G_1 commute with all elements of G_2 and the group law is given by

$$(a_1, a_2)(b_1, b_2) := (a_1 b_1, a_2 b_2) \qquad \forall\, a_i, b_i \in G_i$$

Remarks:

- Proof of group axioms see Lucha & Schöberl
- G_1 and G_2 are normal subgroups of $G_1 \otimes G_2$
- $(g_1, e_2)(e_1, g_2) = (g_1, g_2) = (e_1, g_2)(g_1, e_2)$ elements g_1 and g_2 commute
- $\operatorname{ord} G_1 \otimes G_2 = \operatorname{ord} G_1 \cdot \operatorname{ord} G_2$

Example: $V := C_2 \otimes C_2$ with $V = \{(e_1, e_2), (e_1, d_2), (d_1, e_2), (d_1, d_2)\}$, $d_i^2 = e_i$

V	(e_1, e_2)	(e_1, d_2)	(d_1, e_2)	(d_1, d_2)
(e_1, e_2)	(e_1, e_2)	(e_1, d_2)	(d_1, e_2)	(d_1, d_2)
(e_1, d_2)	(e_1, d_2)	(e_1, e_2)	(d_1, d_2)	(d_1, e_2)
(d_1, e_2)	(d_1, e_2)	(d_1, d_2)	(e_1, e_2)	(e_1, d_2)
(d_1, d_2)	(d_1, d_2)	(d_1, e_2)	(e_1, d_2)	(e_1, e_2)

Compare with $D_2 : e = (e_1, e_2)$, $d = (e_1, d_2)$, $s = (d_1, e_2)$, $sd = (d_1, d_2)$
$\Rightarrow D_2 = C_2 \otimes C_2 \simeq V \Leftrightarrow D_2/C_2 \simeq C_2$
But: $D_3/C_3 \simeq C_2$ does NOT imply $D_3 \simeq C_2 \otimes C_3$ as C_2 is NOT a normal subgroup of D_3. In fact $D_3 \not\simeq C_2 \otimes C_3$. Why?

Semi-direct product: Like the direct product but here elements of G_1 and G_2 do not commute $\Rightarrow$ group law is more complicated.

Euclidean group: Transformations of $\mathbb{R}^3$ consisting of translations $T^3 \simeq \mathbb{R}^3$ and rotations $O(3)$ (including reflection, $R \in O(3)$, $\det R = \pm 1$)

$$E^3 = T^3 \otimes O(3)$$

Poincaré group: Transformations of $\mathbb{R}^4$, equipped with Minkowsky metric, consisting of translations $T^4 \simeq \mathbb{R}^4$ and Lorentz transformations $O(3, 1)$

$$\mathcal{P} = T^4 \otimes O(3, 1)$$

*** End of Tutorial 1 ***