4.6 Symmetries and Group Representations in Quantum Mechanics

4.6.1 Principles of Quantum Mechanics

States:

Physical System Σ	$\Longleftrightarrow$	Hilbert space $\mathcal{H}$ with scalar product $\langle\psi\vert\varphi\rangle$
Pure States in Σ	$\Longleftarrow$	normalised vector $\langle\psi\vert\psi\rangle = 1$ in $\mathcal{H}$

To each unit vector in $\mathcal{H}$ corresponds a pure state in Σ.
However, the reverse is not valid as all elements of

$$|\hat{\psi}\rangle := \{\mathrm{e}^{\mathrm{i}\alpha}|\psi\rangle \in \mathcal{H}, \mathrm{e}^{\mathrm{i}\alpha} \in U(1)\}$$

characterise the same state in Σ. $|\hat{\psi}\rangle$ is called *ray* in $\mathcal{H}$.

Pure states of a quantum mechanical system are rays in Hilbert space $\mathcal{H}$

$$\text{Pure States in } \Sigma \qquad \Longleftrightarrow \qquad \text{ray } |\hat{\psi}\rangle \in \mathcal{H}$$

a vector $|\psi\rangle \in |\hat{\psi}\rangle$ is called representative of ray $|\hat{\psi}\rangle$.
A ray is uniquely defined by one of its representatives.

Transition probability:
The transition probability between two states $|\hat{\psi}\rangle$ and $|\hat{\varphi}\rangle$ is given by

$$\omega(\hat{\psi}, \hat{\varphi}) := |\langle\psi|\varphi\rangle|^2\,, \qquad |\varphi\rangle \in |\hat{\varphi}\rangle\,, \qquad |\psi\rangle \in |\hat{\psi}\rangle$$

Observables:

Measurement	$\Longleftrightarrow$	self-adjoint operators A
Possible measured values	$\Longleftrightarrow$	spectrum of A
Expectation value in state $\vert\hat{\psi}\rangle$	$\Longleftrightarrow$	$\langle\psi\vert A\vert\psi\rangle\,,\quad \vert\psi\rangle \in \vert\hat{\psi}\rangle$

Dynamics:

$$\text{Dynamics = Time evolution} \qquad \Longleftrightarrow \qquad \text{Hamilton operator } H$$

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

Is $\dfrac{\partial H}{\partial t} = 0 \qquad \Rightarrow \qquad |\psi(t)\rangle = \exp\{-(\mathrm{i}/\hbar)H(t-t_0)\}|\psi(t_0)\rangle$
For stationary states $H|\psi_E\rangle = E|\psi_E\rangle$ follows

$$|\psi_E(t)\rangle = \exp\{-(\mathrm{i}/\hbar)E(t-t_0)\}|\psi_E(t_0)\rangle$$

hence

$$|\hat{\psi}_E(t)\rangle = |\hat{\psi}_E(t_0)\rangle$$

4.6.2 Symmetries in Quantum Mechanics

Space-time, geometric symmetries
Passive View:
Fixed system Σ described by two observers with different "coordinates"
Observer 1: $|\hat{\psi}_1\rangle$, $|\hat{\varphi}_1\rangle$,...
Observer 2: $|\hat{\psi}_2\rangle$, $|\hat{\varphi}_2\rangle$,...
Same state described by different rays $|\hat{\psi}_1\rangle$, $|\hat{\psi}_2\rangle$
$\Rightarrow$ there exists a mapping $\hat{T} : \mathcal{H} \to \mathcal{H}$ with $|\hat{\psi}_2\rangle = \hat{T}|\hat{\psi}_1\rangle$

Active View:
Fixed "coordinates" $|\hat{\psi}\rangle$, $|\hat{\varphi}\rangle$,...
System Σ invariant under transformations $\hat{T} : \mathcal{H} \to \mathcal{H}$, that is, invariance of

$$\omega(\hat{\psi}, \hat{\varphi}) := |\langle\hat{\psi}|\hat{\varphi}\rangle|^2 = |\langle\hat{T}\hat{\psi}|\hat{T}\hat{\varphi}\rangle|^2 \qquad \text{and} \qquad [H, \hat{T}] = 0$$

Internal, dynamical symmetries:
Invariance of $\omega(\hat{\psi}, \hat{\varphi})$ and H

Definition:
A invariance or symmetry group is any group of transformations,

$$|\hat{\psi}'\rangle = \hat{T}(g)|\hat{\psi}\rangle\,, \qquad \hat{T}(g_1)\hat{T}(g_2) = \hat{T}(g_1 g_2)$$

which leaves the transition probability $\omega(\hat{\psi}, \hat{\varphi})$ and the Hamiltonian H invariant.

Wigner has shown that such transformations may always be replaced by transformations of the corresponding vectors,i.e. $|\psi'\rangle = U(g)|\psi\rangle$

Wigner theorem:
Let $|\hat{\psi}'\rangle = \hat{T}|\hat{\psi}\rangle$ be a mapping $\hat{T} : \mathcal{H} \to \mathcal{H}$ with $\omega(\hat{\psi}', \hat{\varphi}') = \omega(\hat{\psi}, \hat{\varphi})$, then there exists a mapping $|\psi'\rangle = U|\psi\rangle$ of vectors $\mathcal{H} \to \mathcal{H}$ such that

$$|\psi\rangle \in |\hat{\psi}\rangle \qquad \Rightarrow \qquad |\psi'\rangle \in |\hat{\psi}'\rangle \qquad \forall\, |\psi\rangle \in \mathcal{H}$$

The mapping U has the properties

- $U(|\psi\rangle + |\varphi\rangle) = U|\psi\rangle + U|\varphi\rangle =: |U\psi\rangle + |U\varphi\rangle$
- $U(\lambda|\psi\rangle) = \chi(\lambda)U|\psi\rangle\,, \quad \lambda \in \mathbb{C}$
- $\langle U\psi|U\varphi\rangle = \chi(\langle\psi|\varphi\rangle)$

where either $\chi(\lambda) = \lambda$ or $\chi(\lambda) = \lambda^*$ for all $\lambda \in \mathbb{C}$.

Hence U is either linear and unitary or anti-linear and anti-unitary

Proof: V. Bargmann, J. Math. Phys. **5** (1964) 862-868.

Comments:

- Only invariance of $\omega(\hat{\psi}, \hat{\varphi})$ is required, not that for H!
- The operator U is unique up to a global ($|\hat{\psi}\rangle$-independent) phase. For rays one may replace

$$\hat{T} \to \mathrm{e}^{\mathrm{i}\delta} U$$

For a group of transformations $g \in G$ follows:

$$U(g_1)U(g_2) = \mathrm{e}^{\mathrm{i}\theta(g_1,g_2)} U(g_1 g_2)$$

Such U is called *ray representation* of G in $\mathcal{H}$ (or projective representation). For $\theta(g_1, g_2) = 0$ the ray representation is reduced to the usual often called vector representation.

Conclusions from Wigner theorem:

- Each group of transformations on $\mathcal{H}$ leaving the transition probability $\omega(\hat{\psi}, \hat{\varphi})$ invariant is equivalent to a group of unitary or anti-unitary transformation $U(g)$ forming a ray representation.
- For group elements being continuously connected to the unit element $e \in G$, that is continuous groups, $U(g)$ is always unitary. Hence, finite-dimensional ray representations of simply connected groups are equivalent to vector representations.

4.7 Examples

Invariance under Space Translations

Translation by vector $\vec{a}$: $\vec{r}\,' = T(\vec{a})\vec{r} := \vec{r} + \vec{a}$ with $\vec{a} \in \mathbb{R}^3$

$$T(\vec{a})T(\vec{b}) = \vec{r} + \vec{a} + \vec{b} = T(\vec{a} + \vec{b})$$

Abelian group with 3 parameters.

Ray representation: $T(\vec{a}) \to U(\vec{a})$ with $|\varphi'\rangle = U(\vec{a})|\varphi\rangle$
Position eigenstates: $|\vec{r}\,'\rangle = U(\vec{a})|\vec{r}\rangle := |\vec{r} + \vec{a}\rangle$
Neutral element: $T(\vec{0}) \to U(\vec{0})$
Infinitesimal translation: $U(\delta\vec{a}) = 1 - \frac{\mathrm{i}}{\hbar}\vec{P} \cdot \delta\vec{a}$ with

$$\vec{P} := \mathrm{i}\hbar \frac{\partial U}{\partial \vec{a}}\bigg|_{\vec{a}=\vec{0}}$$

Note: As translations are abelian $\Rightarrow$ $[P_i, P_j] = 0$

Consider

$$U(\vec{a} + \delta\vec{a}) = U(\vec{a})\left(1 - \frac{\mathrm{i}}{\hbar}\vec{P} \cdot \delta\vec{a}\right)$$

$\Rightarrow$

$$U(\vec{a} + \delta\vec{a}) - U(\vec{a}) = -\frac{\mathrm{i}}{\hbar}U(\vec{a})\vec{P} \cdot \delta\vec{a}$$

$\Rightarrow$

$$\frac{\partial U(\vec{a})}{\partial \vec{a}} = -\frac{\mathrm{i}}{\hbar}U(\vec{a})\vec{P}$$

Hence, the operator representing translation in $\mathbb{R}^3$ is unitary and given by

$$\boxed{U(\vec{a}) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{a}}}$$

and $\vec{P}$ can be identified with the momentum operator.
The momentum operator is the generator of translations.

Consider position representation:

$$\begin{aligned}
\langle\vec{r}|(1 - \tfrac{\mathrm{i}}{\hbar}\delta\vec{a} \cdot \vec{P})\psi\rangle &= \langle\vec{r}|U(\delta\vec{a})\psi\rangle \\
&= \langle U(-\delta\vec{a})\vec{r}|\psi\rangle \\
&= \langle\vec{r} - \delta\vec{a}|\psi\rangle \\
&= \psi(\vec{r} - \delta\vec{a}) = \psi(\vec{r}) - \delta\vec{a} \cdot \vec{\nabla}\psi(\vec{r})
\end{aligned}$$

$\Rightarrow$

$$\boxed{\vec{P} = \tfrac{\hbar}{\mathrm{i}}\vec{\nabla}} \quad \text{in position representation}$$

Transformations of operators:

$$A' = U(\vec{a})AU^\dagger(\vec{a}) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{a}}\, A\, \mathrm{e}^{+\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{a}}$$

Momentum operator: $\vec{P}\,' = U(\vec{a})\vec{P}U^\dagger(\vec{a}) = \vec{P}$ is invariant
Position operator: $\vec{R}\,' = U(\vec{a})\vec{R}U^\dagger(\vec{a}) \stackrel{!}{=} \vec{R} - \vec{a}$ follows from $\langle\vec{r}|U(\vec{a})\psi\rangle = \langle\vec{r} - \vec{a}|\psi\rangle$

Consider infinitesimal translation

$$\left(1 - \frac{\mathrm{i}}{\hbar}\vec{P} \cdot \delta\vec{a}\right)\vec{R}\left(1 + \frac{\mathrm{i}}{\hbar}\vec{P} \cdot \delta\vec{a}\right) = \vec{R} - \delta\vec{a}$$

$\Rightarrow$

$$R_j - \frac{\hbar}{\mathrm{i}}\delta a_i[P_i, R_j] = R_j - \delta a_j$$

$\Rightarrow$

$$\boxed{[P_i, R_j] = \tfrac{\hbar}{\mathrm{i}}\,\delta_{ij}}$$

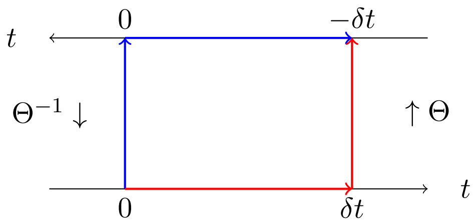
Hamiltonian: $H' = U(\vec{a}) H U^\dagger(\vec{a}) \overset{!}{=} H$ require invariance
$\Rightarrow \quad [H, U(\vec{a})] = 0 \quad \Rightarrow \quad [H, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{a}}] = 0$ for all $\vec{a} \in \mathbb{R}^3$
$\Rightarrow \quad [H, P_j] = 0$
momentum operator is conserved when system is invariant under translations

Time reversal invariance:
Let Θ be the time reversal operator with $\Theta|\psi(t)\rangle := |\psi(-t)\rangle$ obeying

$$|\langle\Theta\psi|\Theta\varphi\rangle|^2 = |\langle\psi|\varphi\rangle|^2$$

Note: $\Theta^2 = 1$, Θ generates a group isomorphic to Z_2

Consider below cycles of time evolution:



Hence

$$\Theta\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\delta t} = \mathrm{e}^{+\frac{\mathrm{i}}{\hbar}H\delta t}\Theta$$

For small δt follows

$$\Theta\left(1 - \tfrac{\mathrm{i}}{\hbar}H\delta t\right)|\psi(0)\rangle = \left(1 + \tfrac{\mathrm{i}}{\hbar}H\delta t\right)\Theta|\psi(0)\rangle$$

$\Rightarrow$

$$-\Theta\mathrm{i}H|\psi(0)\rangle = \mathrm{i}H\Theta|\psi(0)\rangle \qquad \text{for all} \qquad |\psi(0)\rangle \in \mathcal{H}$$

$\Rightarrow$

$$-\Theta\mathrm{i}H = \mathrm{i}H\Theta$$

Case 1: Θ is a linear operator: $\Theta\mathrm{i} = \mathrm{i}\Theta \quad \Rightarrow \quad H\Theta = -\Theta H$
time reversal invariance implies $H = -\Theta H \Theta^{-1} =: -H' = H$
With E also $-E$ is energy eigenvalue $\quad \Rightarrow \quad H$ is not bounded from below ↯

So we are only left with the second option provided by Wigner

Case 2: Θ is anti-linear operator: $\Theta\mathrm{i} = -\mathrm{i}\Theta$

$$\Rightarrow \qquad [\Theta, H] = 0 \qquad \text{as expected}$$

The time reversal operator is an anti-linear operator

Summary:

- Let G be a symmetry group of a given system Σ, $g \in G$.
  Then in most cases the ray representation can be replaced by a unitary vector representation
  $$U(g_1)U(g_2) = U(g_1 g_2) \qquad \text{with} \qquad |\psi\rangle \to |\psi'\rangle = U(g)|\psi\rangle$$
- Properties of the symmetry group
  $$\langle\varphi'|\psi'\rangle = \langle\varphi|U^\dagger(g)U(g)\psi\rangle = \langle\varphi|\psi\rangle$$
  $$H' = U(g)HU^\dagger(g) = H \qquad \Rightarrow \qquad [U(g), H] = 0 \qquad \forall\, g \in G$$

- With $H|\psi_E\rangle = E|\psi_E\rangle$ is also $U(g)|\psi_E\rangle$ eigenvector for same energy eigenvlaue

  $\Rightarrow$ eigenspaces of H are invariant representation spaces for G

  The UIR of symmetry group may be used to classify the spectrum via conserved quantum numbers. With not additional degeneracy each energy eigenvalue corresponds to a UIR according to which the eigenstates are transformed.

4.8 The Lorentz Group

4.8.1 Minkowski space

Consider $\mathbb{R}^4$ with elements

$$x \in \mathbb{R}^4\,, \qquad x = (x^0, x^1, x^2, x^3)^T = (ct, \vec{x})^T = x^\mu$$

equipped with metric

$$g = (g^{\mu\nu}) = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and a scalar product

$$\langle x, y\rangle := x^0 y^0 - \vec{x}\cdot\vec{y} = g_{\mu\nu} x^\mu y^\nu = g^{\mu\nu} x_\mu y_\nu$$

Such a vector space is called *Minkowsky space*

4.8.2 Lorentz transformation

A (homogenous) Lorentz transformation of $\mathbb{R}^4$ is a linear map

$$\Lambda : \begin{array}{l} \mathbb{R}^4 \to \mathbb{R}^4 \\ x \mapsto \Lambda x \end{array} \qquad \text{i.e.} \qquad x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$$

with

$$\langle \Lambda x, \Lambda y\rangle = \langle x, y\rangle \qquad \text{for all} \qquad x, y \in \mathbb{R}^4$$

This implies

$$g_{\mu\nu} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma x^\sigma = g_{\rho\sigma} x^\rho y^\sigma$$

or

$$\Lambda^\mu{}_\rho g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma} \qquad \Leftrightarrow \qquad \Lambda^T g \Lambda = g$$

This results in ten independent equations for the components of Λ like

$$(\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2 = 1$$

Hence for all Λ

$$\Lambda^0{}_0 \geq 1 \qquad \text{or} \qquad \Lambda^0{}_0 \leq -1$$

The composition of two Lorentz transformations is again a Lorentz transformation and

$$\det g = \det(\Lambda^T g \Lambda) = \det g (\det \Lambda)^2 \qquad \Rightarrow \qquad \det \Lambda = \pm 1$$

Hence Λ is invertible and the set of all Lorentz transformations forms a group called *Lorentz group* denoted by

$$\mathcal{L} = O(3,1)$$

Only six of the sixteen elements of Λ can be chosen independently, that is, $\dim \mathcal{L} = 6$.

Rotations:
Let $R(\vec{\omega}) \in SO(3)$ be a 3×3-rotation matrix acting in $\mathbb{R}^3$ then

$$\Lambda(\vec{\omega}) = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & R(\vec{\omega}) \end{pmatrix}$$

is a Lorentz transformation. Actually $SO(3) \subset O(3,1)$.

Boosts:
Let $\vec{\beta} \in \mathbb{R}$ with $\beta := |\vec{\beta}| \leq 1$ (note that $\vec{\beta} = \vec{v}/c$) and

$$\gamma(\beta) := \frac{1}{\sqrt{1-\beta^2}}$$

then

$$\Lambda(\vec{\beta}) = \begin{pmatrix} \gamma(\beta) & \gamma(\beta)\vec{\beta}^T \\ \gamma(\beta)\vec{\beta} & 1_3 + \frac{\gamma(\beta)-1}{\beta^2}\vec{\beta}\vec{\beta}^T \end{pmatrix}$$

is a Lorentz transformation.
NOTE: The set of boost does NOT form a group, but the subset of boost into the same direction forms a subgroup. (Proof!!!)
Let $\vec{\beta} = (\beta, 0, 0)^T$ and $\gamma(\beta) = \gamma$ then

$$\Lambda(\vec{\beta}) = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \Lambda(\vec{\beta})x = \begin{pmatrix} \gamma c(t+vx) \\ \gamma(x^1+vt) \\ x^2 \\ x^3 \end{pmatrix}$$

with $\tanh\theta := \beta$, $\theta \in \mathbb{R}$ is called the *rapidity*, we find

$$\Lambda(\vec{\beta}) = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \in SO(1,1)$$

Reflections:

$$P := \begin{pmatrix} 1 & 0 \\ 0 & -1_3 \end{pmatrix}$$

Time reversal:

$$T := \begin{pmatrix} -1 & 0 \\ 0 & 1_3 \end{pmatrix}$$

Connected Components of $\mathcal{L} = O(3,1)$:

$$\begin{aligned}
\mathcal{L}_+^\uparrow &:= \left\{\Lambda \in \mathcal{L} | {\Lambda^0}_0 \geq 1\,, \det\Lambda = 1\right\} && \text{proper LT} \\
\mathcal{L}_-^\uparrow &:= \left\{\Lambda \in \mathcal{L} | {\Lambda^0}_0 \geq 1\,, \det\Lambda = -1\right\} = P\mathcal{L}_+^\uparrow \\
\mathcal{L}_-^\downarrow &:= \left\{\Lambda \in \mathcal{L} | {\Lambda^0}_0 \leq -1\,, \det\Lambda = -1\right\} = T\mathcal{L}_+^\uparrow \\
\mathcal{L}_+^\downarrow &:= \left\{\Lambda \in \mathcal{L} | {\Lambda^0}_0 \leq -1\,, \det\Lambda = 1\right\} = PT\mathcal{L}_+^\uparrow
\end{aligned}$$

Any $\Lambda \in \mathcal{L}_+^\uparrow$ can be put into the form $\Lambda = \Lambda(\vec{\beta})\Lambda(\vec{\omega})$
$SO(3,1) = \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\downarrow$ and $\mathcal{L}^\uparrow = \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\uparrow$ are subgroups of $\mathcal{L} = O(3,1)$

4.8.3 The $SO(3,1)$ algebra

The generators of the Lorentz group are obtained from the neighborhood of the unit element belonging to $\mathcal{L}_+^\uparrow$.
Those for rotations are given by

$$J_i = \begin{pmatrix} 0 & 0 \\ 0 & L_i \end{pmatrix} \qquad \text{where } L_i \text{ are the } 3\times 3 \text{ as given in Homework problem 10}$$

Explicitly, they read

$$J_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-\mathrm{i}\\0&0&\mathrm{i}&0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&\mathrm{i}\\0&0&0&0\\0&-\mathrm{i}&0&0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0&0&0&0\\0&0&-\mathrm{i}&0\\0&\mathrm{i}&0&0\\0&0&0&0 \end{pmatrix}.$$

For the boost these generator read

$$K_1 = \begin{pmatrix} 0&-\mathrm{i}&0&0\\-\mathrm{i}&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0&0&-\mathrm{i}&0\\0&0&0&0\\-\mathrm{i}&0&0&0\\0&0&0&0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0&0&0&-\mathrm{i}\\0&0&0&0\\0&0&0&0\\-\mathrm{i}&0&0&0 \end{pmatrix}.$$

Note that, for example

$$J_1^2 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \qquad K_1^2 = -\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},$$

and hence

$$\mathrm{e}^{\mathrm{i}\omega J_1} = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&\cos\omega&\sin\omega\\0&0&-\sin\omega&\cos\omega \end{pmatrix}, \qquad \mathrm{e}^{\mathrm{i}\beta K_1} = \begin{pmatrix} \cosh\beta&\sinh\beta&0&0\\\sinh\beta&\cosh\beta&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},$$

An arbitrary element of $\mathcal{L}_+^\uparrow$ is the generated by $\Lambda(\vec{\omega},\vec{\beta}) = \mathrm{e}^{\mathrm{i}(\vec{\omega}\cdot\vec{J}+\vec{\beta}\cdot\vec{K})}$ plus the P and T matrices.

The generators obey the $so(3,1)$-algebra

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k\,, \qquad [J_i, K_j] = \mathrm{i}\varepsilon_{ijk}K_k\,, \qquad [K_i, K_j] = -\mathrm{i}\varepsilon_{ijk}J_k$$

The group $SO(3,1)$ is locally isomorphic to $SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in \mathbb{C}, ad-bc=1 \right\}$.

Let $J_i^{(\pm)} = \frac{1}{2}(J_i \pm \mathrm{i}K_i)$ then

$$[J_i^{(\pm)}, J_j^{(\pm)}] = \mathrm{i}\varepsilon_{ijk}J_k^{(\pm)} \qquad \text{and} \qquad [J^{(+)}{}_i, J^{(-)}{}_j] = 0$$

That implies $so(3,1) \to so(4) = so(3) \oplus so(3)$.
Hence the representations are given by the set (j_+, j_-) with $j_\pm = 0, \frac{1}{2}, 1, \ldots$.
With $\vec{J} = \vec{J}^{(+)} + \vec{J}^{(-)}$ and $\vec{K} = -\mathrm{i}(\vec{J}^{(+)} - \vec{J}^{(-)})$ the representation matrices are given by

$$D^{(j_+,j_-)}(\vec{\omega},\vec{\beta}) = \exp\left\{\mathrm{i}(\vec{\omega}-\mathrm{i}\vec{\beta})\cdot\vec{J}^{(+)} + \mathrm{i}(\vec{\omega}+\mathrm{i}\vec{\beta})\cdot\vec{J}^{(-)}\right\}$$

Example: Spinor representation $(\frac{1}{2}, 0)$

$$D^{(\frac{1}{2},0)}(\omega\vec{e}_3,\vec{0}) = \mathrm{e}^{\mathrm{i}\omega\frac{\sigma_3}{2}} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\omega}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\omega}{2}} \end{pmatrix}, \qquad D^{(\frac{1}{2},0)}(\vec{0},\beta\vec{e}_3) = \mathrm{e}^{\beta\frac{\sigma_3}{2}} = \begin{pmatrix} \mathrm{e}^{\frac{\beta}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\beta}{2}} \end{pmatrix}$$

More later.

4.9 The Poincaré Group

Poincaré transformation:
The Poincaré transformation $\Pi = (a, \Lambda)$, with $\Lambda \in \mathcal{L}$ and $a \in \mathbb{R}^4$, is a mapping defined by

$$\Pi : \begin{array}{l} \mathbb{R}^4 \to \mathbb{R}^4 \\ x \mapsto \Pi(x) = \Lambda x + a \end{array}$$

The set of all Poincaré transformations forms the *Poincaré group* $\mathcal{P}$ with composition law

$$(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2)$$

and therefore is the semi-direct product of the Lorentz group and the group of space-time translations in Minkowsky space

$$\mathcal{P} = \mathbb{R}^4 \rtimes \mathcal{L}$$

Generators: Acting on $L^2(\mathbb{R}^4)$

- Time translations: $H = \mathrm{i}\partial_0$
- Space translations: $\vec{P} = -\mathrm{i}\vec{\nabla}$
- Rotations: $\vec{J} = -\mathrm{i}(\vec{r} \times \vec{\nabla})$
- Boosts: $\vec{K} = -\mathrm{i}(\vec{r}\partial_0 + x_0\vec{\nabla})$

$\Rightarrow$ 10 parameter group

Algebra:

$$\left.\begin{array}{l} \left.\begin{array}{ll} [J_i, J_j] = \mathrm{i}\varepsilon_{ijk} J_k & so(3) \text{ algebra} \\ [K_i, K_j] = -\mathrm{i}\varepsilon_{ijk} J_k & \\ [J_i, K_j] = \mathrm{i}\varepsilon_{ijk} K_k & \end{array}\right\} so(3,1) \text{ algebra} \\ [H, P_i] = [P_i, P_j] = [H, J_i] = 0 \\ [H, \vec{K}] = \mathrm{i}\vec{P} \\ [P_k, K_l] = \mathrm{i}H\delta_{kl} \\ [J_i, P_j] = \mathrm{i}\varepsilon_{ijk} P_k \end{array}\right\} \text{Poincarè algebra}$$

In covariant notation

$$P^\mu := (H, \vec{P}) \,, \qquad M^{\mu\nu} := x^\mu P^\nu - x^\nu P^\mu$$

the algebra reads

$$\begin{aligned} &[M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}\,(M^{\mu\sigma} g^{\nu\rho} - M^{\nu\sigma} g^{\mu\rho} + M^{\mu\rho} g^{\nu\sigma} - M^{\nu\rho} g^{\mu\sigma}) \\ &[M^{\mu\nu}, P^\rho] = \mathrm{i}\,(g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu) \\ &[P^\mu, P^\nu] = 0 \end{aligned}$$

Note following relations:

$$\begin{aligned} &M^{0i} = K^i \,, \qquad M^{ij} = \varepsilon^{ijk} J_k \,, \qquad M^{\mu\nu} = -M^{\nu\mu} \,, \\ &\tfrac{1}{2} M^{\mu\nu} M_{\mu\nu} = \vec{J}^2 - \vec{K}^2 \,, \qquad M^{\mu\nu} M^{\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} = \vec{J} \cdot \vec{K} \end{aligned}$$

Casimirs: $\mathcal{P}$ has two Casimir operators

- Mass: $P^\mu P_\mu = m^2 \mathbf{1}$
- Generalised spin: $W^\mu W_\mu = w^2 \mathbf{1}$

Pauli-Lubanski operator:

$$W_\mu := -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma$$

In rest system $P^\mu = (m, \vec{0})$ follows

$$W^\mu = (\vec{J}\cdot\vec{P}, H\vec{J} - \vec{K}\times\vec{P}) = (0, m\vec{J}) = m(0, \vec{J})$$

Hence

$$w^2 = -m^2 s(s+1)$$

That is, elementary particles are uniquely defined by their mass m and spin s. The associated states transform under a certain (j_+, j_-)-representation of $so(3,1)$ which is finite-dimensional but non-unitary. Any additional properties like charge, color, etc. are realised by internal symmetries like gauge-invariance (Coleman-Mandula NoGo-Theorem).

4.10 Irreducible Representations of Lorentz Group

The Lorentz group is not compact. As a result the irreducible representations can be classified in one of two possibilities.

- Finite dimensional but non-unitary representation.
- Infinite dimensional but unitary representation.

This is a deep statement and has far reaching consequences in the study of relativistic quantum mechanics. In quantum field theories the field operators transform under certain finite dimensional representations, while for the consistency of quantum mechanics, the states must transform under certain unitary representation.

Finite-dimensional representations

Recall $J_i^{(\pm)} = \frac{1}{2}(J_i \pm \mathrm{i}K_i) \quad \Rightarrow \quad (J_i^{(\pm)})^\dagger = J_i^{(\mp)}$

Furthermore under parity transformation P: $\vec{J} \to \vec{J}$ but $\vec{K} \to -\vec{K} \quad \Rightarrow \quad J_i^{(\pm)} \to J_i^{(\mp)}$

Hence for the reps $(j_+, j_-) \;\Rightarrow\; (j_-, j_+) \;\Rightarrow\;$ only (j,j) reps are invariant under parity.

Or use parity doubling $(j_+, j_-) \oplus (j_-, j_+)$

Examples:

- $(0,0)$: Lorentz scalars (1-dim. trivial reps)
- $(\frac{1}{2}, 0)$: Right Weyl spinor (2-dim. parity is broken)
- $(0, \frac{1}{2})$: Left Weyl spinor (2-dim. parity is broken)
- $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$: Dirac spinor (4-dim. parity doubling)
- $(\frac{1}{2}, \frac{1}{2})$: 4-vector reps (4-dim. fundamental reps, parity conserved)
- $(1,0) \oplus (0,1)$: Anti-sym. two-tensor, e.g. $F^{\mu\nu}$ (6-dim. parity doubling)

4.11 Irreducible Representations of Poincaré Group

Eigenstates of Casimir $P^\mu P_\mu$ and $W^\mu W_\mu$ span the representation space

$$P^\mu P_\mu \left| \begin{matrix} m & w \\ \vec{p} & w_3 \end{matrix} \right\rangle = m^2 \left| \begin{matrix} m & w \\ \vec{p} & w_3 \end{matrix} \right\rangle , \qquad W^\mu W_\mu \left| \begin{matrix} m & w \\ \vec{p} & w_3 \end{matrix} \right\rangle = w^2 \left| \begin{matrix} m & w \\ \vec{p} & w_3 \end{matrix} \right\rangle$$

Pair of eigenvalues (m, w) define reps space and $(\vec{p}, w_3)$ enumerate the basis vectors

In rest system $W^\mu W_\mu = -m^2\vec{J}^2$ and $W_3 = -mJ_3$. Hence

$$W^\mu W_\mu \left| \begin{matrix} m & w \\ \vec{0} & w_3 \end{matrix} \right\rangle = -m^2 s(s+1) \left| \begin{matrix} m & w \\ \vec{0} & w_3 \end{matrix} \right\rangle , \qquad W_3 \left| \begin{matrix} m & w \\ \vec{0} & w_3 \end{matrix} \right\rangle = -m s_3 \left| \begin{matrix} m & w \\ \vec{0} & w_3 \end{matrix} \right\rangle$$

Wigner states:
A particle at rest is uniquely characterised by its mass m and spin s. A state with $\vec{p} \neq \vec{0}$ can be generated via a Lorentz boost $L(p)$ with $p^\mu = L(p)^\mu{}_\nu k^\nu$, $k^\nu = (m, \vec{0})$.
We my change notation $w \to s$ and $w_3 \to s_3$ to define the Wigner states:

$$\left| \begin{matrix} m & s \\ \vec{p} & s_3 \end{matrix} \right\rangle = U(L(p)) \left| \begin{matrix} m & s \\ \vec{0} & s_3 \end{matrix} \right\rangle$$

Properties:

$$U(\Lambda) \left| \begin{matrix} m & s \\ \vec{p} & s_3 \end{matrix} \right\rangle = \sum_{s_3'=-s}^{s} D^s_{s_3' \, s_3}(R(\Lambda, p)) \left| \begin{matrix} m & s \\ \Lambda\vec{p} & s_3' \end{matrix} \right\rangle$$

$$U(a) \left| \begin{matrix} m & s \\ \vec{p} & s_3 \end{matrix} \right\rangle = \mathrm{e}^{-\mathrm{i} p^\mu a_\mu} \left| \begin{matrix} m & s \\ \vec{p} & s_3 \end{matrix} \right\rangle$$

Here $D^s(R)$ is the $2s+1$-dim. matrix representing the $SU(2)$-Wigner rotation R
Wigner rotation $R(\Lambda, p) := L^{-1}(\Lambda p)\Lambda L(p)$ is a pure rotation

Orthogonality:

$$\left\langle \begin{matrix} m & s \\ \vec{p} & s_3 \end{matrix} \middle| \begin{matrix} m' & s' \\ \vec{p}' & s_3' \end{matrix} \right\rangle = \delta_{mm'}\delta_{ss'}\delta(\vec{p} - \vec{p}')\delta_{s_3 s_3'} \, 2\sqrt{m^2 + \vec{p}^2}$$

Lit.: S. Weinberg, *The Quantum Theory of Fields I + II* (CUP, 1995)