

3.5 Harmonic Analysis on Homogenous Spaces

Let $f(g) \in L^2(G/H)$, H massive with $H\vec{x}_0 = \vec{x}_0$

$\Rightarrow f(\vec{x}) = f(g\vec{x}_0) =: \hat{f}(g) \in L^2(G)$ such that $\hat{f}(gh) = \hat{f}(g)$ for all $h \in H$.

All $f(g) \in L^2(G/H)$ are associate spherical functions.

Peter-Weyl Theorem: continuous version

$$\begin{aligned} \hat{f}(g) &= \sum_{\text{all UIR } j \text{ of } G} d_j \sum_{m,n} \tilde{f}_{nm}^j D_{mn}^j(g) \\ \tilde{f}_{nm}^j &= \int_G dg \hat{f}(g) D_{mn}^{j*}(g) \end{aligned}$$

Fourier coefficient for associate spherical functions:

$$\tilde{f}_{nm}^j = \int_G dg \hat{f}(g) D_{mn}^{j*}(g) = \int_G dg \hat{f}(g) D_{mn}^{j*}(gh) = \int_G dg \hat{f}(g) \sum_k D_{mk}^j(g) D_{kn}^{j*}(h)$$

Integration over h :

$$\tilde{f}_{nm}^j = \int_G dg \hat{f}(g) D_{mn}^{j*}(g) = \int_G dg \hat{f}(g) D_{mn}^{j*}(gh) = \int_G dg \hat{f}(g) \sum_k D_{mk}^j(g) \int_H dh D_{kn}^{j*}(h)$$

Note: The UIR D^j of G is in general a reducible unitary reps of $H \subset G$. So let's decompose into UIR D^α of H

$$D^j = \sum_{\alpha} c_{\alpha} D^{\alpha}$$

1) Let $j \neq \text{class 1}$: No invariant vector in $D^j \Rightarrow D^j$ does not contain trivial representation

$$\int_H dh D_{kn}^{j*}(h) = 0 \quad \text{for all } j \neq \text{class 1}$$

2) Let $j = \text{class 1}$: As H is massive \Rightarrow exists only one vector $|\varphi_0\rangle$ such that $D^j(h)|\varphi_0\rangle = |\varphi_0\rangle$. That is, $D_{00}^j(h) = \langle\varphi_0|D^j(h)|\varphi_0\rangle = 1$.

In other words, the trivial reps of H appears exactly once in above decomposition and

$$\int_H dh D_{kn}^{j*}(h) = \delta_{k0}\delta_{n0} \quad \text{for all } j = \text{class 1}$$

Result:

$$\tilde{f}_{nm}^j = \int_G dg \hat{f}(g) D_{m0}^j(g) \delta_{n0}$$

Let Λ be the set of class 1 reps of G :

$$\begin{aligned} f(\vec{x}) = \hat{f}(g) &= \sum_{j \in \Lambda} d_j \sum_{m=0}^{d_j-1} \tilde{f}_m^j D_{m0}^j(g) \\ \tilde{f}_m^j &= \int_G dg \hat{f}(g) D_{m0}^{j*}(g) \end{aligned}$$

Generalised spherical harmonics:

$$Y_{\ell m}(\vec{x}) := \sqrt{d_{\ell}} D_{m0}^{\ell}(g), \quad \vec{x} = g\vec{x}_0$$

Form complete orthonormal set on $L^2(G/H)$:

$$\int_{G/H} d\mu(\vec{x}) Y_{\ell m}(\vec{x}) Y_{\ell' m'}^*(\vec{x}) = \int_G dg \sqrt{d_{\ell} d_{\ell'}} \underbrace{D_{m0}^{\ell}(g) D_{m'0}^{\ell'*}(g)}_{\delta'_{\ell\ell} \delta_{mm'} / d_{\ell}} = \delta'_{\ell\ell} \delta_{mm'}$$

Harmonic analysis on homogenous spaces G/H (with massive H):

$$\boxed{\begin{aligned} f(\vec{x}) &= \sum_{\ell \in \Lambda} \sum_m c_{\ell m} Y_{\ell m}(\vec{x}) \\ c_{\ell m} &= \int_{G/H} d\mu(\vec{x}) f(\vec{x}) Y_{\ell m}^*(\vec{x}) \end{aligned}}$$

Note: $Y_{\ell m}(\vec{x})$ are also eigenfunctions of Laplace-Beltrami operator on $\mathcal{M} = G/H$.

Harmonic analysis for zonal spherical functions: $f(h_1^{-1}gh_2) = f(g)$, $h_1, h_2 \in H$

$$\boxed{\begin{aligned} f(g) &= \sum_{\ell \in \Lambda} d_\ell \lambda_\ell D_{00}^\ell(g) \\ \lambda_\ell &= \int_G dg f(g) D_{00}^{\ell*}(g) \end{aligned}}$$

3.6 Generators of Lie Groups

Let G be a Lie group with $g = g(\alpha) \in G$, $\alpha = (\alpha^1, \dots, \alpha^n)$, $e = g(0)$ neutral element

Consider reps of G in some \mathcal{H} : $D(g) = D(g(\alpha))$ with $D(g(0)) = 1$

Generator:

$$X_a := \left. \frac{\partial D(g(\alpha))}{\partial \alpha^a} \right|_{\alpha=0}$$

In quantum mechanics often $T_a = \pm iX_a$ as they are self-adjoint operators for unitary reps

Generators are in essence the reps matrices near neutral element as

$$D(g(\delta\alpha)) = 1 + \delta\alpha^a X_a + O(\delta\alpha^2)$$

$$D(g^{-1}(\delta\alpha)) = [D(g(\delta\alpha))]^{-1} = 1 - \delta\alpha^a X_a + O(\delta\alpha^2)$$

We now use summation convention, so in above sum over a

Comments:

- Generators depend on parametrisation of G
- For unitary reps $D(g^{-1}(\delta\alpha)) = D^\dagger(g(\delta\alpha)) \Rightarrow X_a^\dagger = -X_a$ or $T_a^\dagger = T_a$
- For $\dim \mathcal{H} = d < \infty$: X_a or T_a are $d \times d$ matrices
- For $\dim \mathcal{H} = \infty$: X_a or T_a are linear operators acting on vectors in \mathcal{H}

Consider:

$$\begin{aligned} D(g(\delta\gamma)) &= D(g^{-1}(\beta)g(\delta\alpha)g(\beta)) \\ &= 1 + \delta\alpha^a D(g^{-1}(\beta))X_a D(g(\beta)) \\ &= 1 + \delta\gamma^a X_a \end{aligned}$$

That is $D(g^{-1}(\beta))X_a D(g(\beta))$ is a linear combination of generators

Now $\beta \rightarrow \delta\beta$ small

$$(1 - \delta\beta^b X_b)X_a(1 + \delta\beta^b X_b) = X_a + \delta\beta^b (X_a X_b - X_b X_a) + O(\delta\beta^2)$$

\Rightarrow

$$\boxed{[X_a, X_b] := (X_a X_b - X_b X_a) = c_{ab}^d X_d}$$

The constants c_{ab}^d are called *structure constants*

Properties

- Depend on parametrisation of G
- $c_{ab}^d = -c_{ba}^d$ antisymmetric in lower indices, $c_{aa}^d = 0$
- From Jacobi identity $[[X_a, X_b], X_c] + [[X_b, X_c], X_a] + [[X_c, X_a], X_b] = 0$ follows

$$c_{ab}^d c_{dc}^e + c_{bc}^d c_{da}^e + c_{ca}^d c_{db}^e = 0$$

- G abelian $\Leftrightarrow c_{ab}^d = 0$
Obvious as $D(g^{-1}(\beta))X_a D(g(\beta)) = X_a \Leftrightarrow [X_a, X_b] = 0$ for all a, b, d

Example: $G = SO(3)$ (see Homework 4 Problem 10 with $\hbar = 1$)

$$[L_i, L_j] = i\varepsilon_{ijk} L_k \quad \Rightarrow \quad c_{ij}^k = i\varepsilon_{ijk}$$

3.7 Generators for transformation groups

Consider transformation on \mathcal{M} : $x' = g(\alpha)x$ and let $\alpha \rightarrow \delta\alpha$
Then the μ -component of x'_μ is given by

$$x'_\mu = x_\mu + \delta x_\mu = x_\mu + \delta\alpha^a U_{a\mu}(x) + O(\delta^2\alpha)$$

where

$$U_{a\mu}(x) := \frac{\delta x_\mu}{\delta\alpha^a}.$$

Consider reps in $\mathcal{H} = L^2(\mathcal{M})$:

$$(D(g(\alpha))\psi)(x) = \psi(g^{-1}(\alpha)x)$$

then again let $\alpha \rightarrow \delta\alpha$

$$(\psi + \delta\alpha^a X_a \psi)(x) = \psi(x - \delta x) = \psi(x) - \delta\alpha^a U_{a\mu}(x) \frac{\partial \psi}{\partial x_\mu}(x)$$

Generator: Is a linear differential operator on \mathcal{H}

$$X_a = -U_{a\mu}(x) \frac{\partial}{\partial x_\mu} = -U_{a\mu}(x) \partial^\mu$$

Example: $\mathcal{M} = \mathbb{R}^2$, $G = SO(2)$

$$\begin{aligned} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} &= \begin{pmatrix} \cos(\delta\alpha) & -\sin(\delta\alpha) \\ \sin(\delta\alpha) & \cos(\delta\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} 1 & -\delta\alpha \\ \delta\alpha & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \delta\alpha \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \end{aligned}$$

Hence $U_1(x) = \frac{\delta x_1}{\delta\alpha} = -x_2$ and $U_2(x) = \frac{\delta x_2}{\delta\alpha} = x_1$
 \Rightarrow

$$X = -U_\mu(x) \partial^\mu = -(-x_2 \partial^1 + x_1 \partial^2) = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

Or with $T = i\hbar X$

$$T = i\hbar \left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \right) = L_3$$

Rotations in the plane are generated by the angular momentum operator L_3 on $L^2(\mathbb{R}^2)$.

Generator of rotation matrix (2-dim reps in \mathbb{R}^2)

$$\begin{pmatrix} \cos(\delta\alpha) & -\sin(\delta\alpha) \\ \sin(\delta\alpha) & \cos(\delta\alpha) \end{pmatrix} \approx 1 + \delta\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 1 - i\delta\alpha \sigma_2$$

Generator for rotation matrix in \mathbb{R}^2 is given by Pauli matrix $X = -i\sigma_2$
Finite rotation by angle α is given

$$\begin{aligned} e^{\alpha X} &= \sum_{n=0}^{\infty} \frac{1}{n!} \alpha^n X^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (-i\alpha)^n \sigma_2^n \quad (\sigma_2^2 = 1) \\ &= \mathbf{1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \alpha^{2k} - i\sigma_2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \alpha^{2k+1} \\ &= \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} = e^{-i\alpha \sigma_2} \end{aligned}$$

Finite group elements can be represented by generators via exponentiation.

For each generator exists an one-parameter subgroup represented by

$$D(g(t)) = e^{tX_t}$$

Recall with $\delta t = t/n$ for large n

$$D(g(t)) = [D(g(\delta t))]^n = \lim_{n \rightarrow \infty} [1 + \delta t X_t]^n = \lim_{n \rightarrow \infty} \left[1 + \frac{t}{n} X_t \right]^n = e^{tX_t}$$

4 Lie Algebras

4.1 Definitions

A finite-dimensional vector space \mathcal{L} (real or complex) is called *Lie Algebra* if there exists a *Lie Product* \circ such that

- $X \circ Y \in \mathcal{L} \quad \forall X, Y \in \mathcal{L}$
- $(\alpha X + \beta Y) \circ Z = \alpha X \circ Y + \beta Y \circ Z \quad \forall X, Y, Z \in \mathcal{L}, \quad \alpha, \beta \in \mathbb{R} \text{ or } \mathbb{C}$
- $Z \circ Y = -Y \circ X \quad \text{antisymmetric}$
- $X \circ (Y \circ Z) + Y \circ (Z \circ X) + Z \circ (X \circ Y) = 0 \quad \text{Jacobi Identity}$

Example: $\mathcal{L} = \mathbb{R}^3$ with $\vec{x} \circ \vec{y} := \vec{x} \times \vec{y}$

Theorem: The generators of a Lie Group form a Lie Algebra with

$$X \circ Y := [X, Y] = XY - YX$$

Comments:

- Notation $G = SO(n) \Leftrightarrow \mathcal{L} = so(n)$, that is, use lower case notation for algebra
- *locally isomorphic* groups $:\Leftrightarrow$ have the same (isomorphic) Lie Algebra
Example: $so(3) \simeq su(2)$ are isomorphic
- Reps of group \Leftrightarrow Reps of algebra
- $\dim G = \dim \mathcal{L}$

From now on we consider only

$$X \circ Y = [X, Y]$$

Homomorphism:

$$T : \begin{array}{l} \mathcal{L} \rightarrow \mathcal{L}' \\ X \mapsto T(X) \end{array}$$

such that $T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$ and $T([X, Y]) = [T(X), T(Y)]$

Subalgebra: Subspace $\mathcal{N} \subset \mathcal{L}$ with $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$

Ideal: Subalgebra $\mathcal{N} \subset \mathcal{L}$ with $[\mathcal{L}, \mathcal{N}] \subset \mathcal{N}$

trivial ideals are $\mathcal{L} = \{0\}$ and $\mathcal{N} = \mathcal{L}$, rest are called proper ideals

Center: max. ideal such that $[\mathcal{N}, \mathcal{N}] = 0$ abelian algebra

Simple Lie algebra: \mathcal{L} has no proper ideal

Semi-simple Lie Algebra: \mathcal{L} has no abelian ideals

4.2 Representations of Lie Algebras

Homomorphism:

$$D : \begin{array}{l} \mathcal{L} \rightarrow \text{lin. operators on } \mathcal{D} \\ X \mapsto D(X) \end{array}$$

is called *representation* of \mathcal{L} in \mathcal{D}

Reps of Lie Group	\Leftrightarrow	Reps of Lie Algebra
$D(g(\alpha)) = e^{\alpha^a X_a}$		X_a
$D(g(\alpha)) = e^{i\alpha^a T_a}$		T_a
unitary		$X_a^\dagger = -X_a, T_a^\dagger = T_a$
irreducible		irreducible

Adjoint Representation:

Let $X \in \mathcal{L}$ be fixed

$$ad(X) : \begin{array}{l} \mathcal{L} \rightarrow \mathcal{L} \\ Y \mapsto [X, Y] \end{array}$$

$$\begin{aligned} [ad(X), ad(Y)]Z &= ad(X)ad(Y)Z - ad(Y)ad(X)Z \\ &= ad(X)[Y, Z] - ad(Y)[X, Z] \\ &= [X, [Y, Z]] - [Y, [X, Z]] \\ &= -[Z, [X, Y]] \quad \text{Jacobi Id.} \\ &= [[X, Y], Z] \\ &= ad([X, Y])Z \quad \forall Z \in \mathcal{L} \end{aligned}$$

Let X_1, \dots, X_n be basis in \mathcal{L} with

$$[X_i, X_k] = c_{ik}^l X_l$$

then

$$ad(X_i)X_k = [X_i, X_k] = c_{ik}^l X_l$$

The structure constants are the matrix elements of the adjoint representation

$$ad(X_i)_k^l = c_{ik}^l$$

Note: $\dim ad(X) = \dim \mathcal{L} = \dim G$

Example $so(3)$: $\dim so(3) = 3$ (see Homework 4 Problem 10)

4.3 Cartan Metric

Scalar product in \mathcal{L} : (Killing form)

$$(X, Y) := \text{Tr}(ad(X)ad(Y))$$

Using above basis X_1, \dots, X_n in \mathcal{L}

Cartan metric:

$$g_{kl} := \text{Tr}(ad(X_k)ad(X_l)) = c_{kr}^s c_{ls}^r = g_{lk} \quad \text{symmetric}$$

Let $X = a^k X_k$ and $Y = b^l X_l$ then

$$(X, Y) = \text{Tr}(a^k ad(X_k) b^l ad(X_l)) = \text{Tr}(ad(X_k) ad(X_l)) a^k b^l = g_{kl} a^k b^l$$

Cartan Criterium:

$$\det(g_{kl}) \neq 0 \quad \Leftrightarrow \quad \mathcal{L} \text{ semi-simple}$$

This implies the existence of the inverse metric tensor: $g_{kl} g^{lm} = \delta_k^m$

Lie algebra *compact* $\Leftrightarrow g_{kl}$ positive or negative definite

Examples:

- $so(3)$: $[X_i, X_j] = \varepsilon_{ijk} X_k \Rightarrow g_{kl} = \varepsilon_{krs} \varepsilon_{lrs} = 2\delta_{kl}$
semi-simple and compact
also with $L_i = iX_i \Rightarrow [L_i, L_j] = i\varepsilon_{ijk} L_k \Rightarrow g_{kl} = -2\delta_{kl}$

- $so(2, 1)$: $[X_1, X_2] = X_3, [X_2, X_3] = -X_1, [X_3, X_1] = X_2$
 $\Rightarrow c_{12}^3 = 1 = -c_{21}^3, c_{23}^1 = -1 = -c_{32}^1, c_{31}^2 = 1 = -c_{13}^2$
 $\Rightarrow g_{11} = \text{Tr}(ad(X_1)ad(X_1)) = 2c_{12}^3 c_{13}^2 = -2$
 $\Rightarrow g_{22} = \text{Tr}(ad(X_2)ad(X_2)) = 2c_{23}^1 c_{21}^3 = 2$
 $\Rightarrow g_{33} = \text{Tr}(ad(X_3)ad(X_3)) = 2c_{31}^2 c_{32}^1 = 2$

$$\Rightarrow (g_{kl}) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{semi-simple and NOT compact}$$

$$\text{Let } T_i = iX_i \Rightarrow [T_1, T_2] = iT_3, [T_2, T_3] = -iT_1, [T_3, T_1] = -iT_2$$

$$\Rightarrow (g_{kl}) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

This $so(2, 1) \simeq su(1, 1)$ algebra is often used in QM as *spectrum generating algebra*
 T_3 is called the compact operator

Consider:

$$c_{ijk} := g_{il} c_{jk}^l = c_{ir}^s c_{ls}^r c_{jk}^l = c_{ir}^s \left(-c_{ks}^l c_{lj}^r - c_{sj}^l c_{lk}^r \right) = c_{ir}^s c_{jl}^r c_{ks}^l + c_{ri}^s c_{sj}^l c_{lk}^r$$

$\Rightarrow c_{ijk}$ is totally anti-symmetric

4.4 The Casimir Operator

Definition:

$$C := g^{kl} X_k X_l = g_{kl} X^k X^l$$

Consider

$$\begin{aligned} [C, X_i] &= g^{kl} [X_k X_l, X_i] \\ &= g^{kl} X_k [X_l, X_i] + g^{kl} [X_k, X_i] X_l \\ &= g^{kl} X_k c_{li}^r X_r + g^{kl} c_{ki}^r X_r X_l \\ &= g^{kl} X_k c_{li}^r X_r + g^{kl} c_{li}^r X_r X_k \quad \text{as } g^{kl} = g^{lk} \\ &= g^{kl} c_{li}^r (X_k X_r + X_r X_k) \\ &= g^{kl} g^{rs} \underbrace{(X_k X_r + X_r X_k)}_{\text{sym. } k \leftrightarrow r} \underbrace{c_{lis}}_{\text{antisym. } l \leftrightarrow s} = 0 \\ &\quad \underbrace{\hspace{10em}}_{\text{sym. } l \leftrightarrow s} \end{aligned}$$

The Casimir operator commutes with all elements of \mathcal{L}

In case of an irreducible reps follows via Schur lemma: $C = \lambda \mathbf{1}$

These eigenvalues are used to characterise the irreducible reps

Examples:

- $so(3) \simeq su(2)$: $g_{kl} = -2\delta_{kl}, g^{kl} = -\frac{1}{2}\delta^{kl}$
 $C = -\frac{1}{2}(X_1^2 + X_2^2 + X_3^2) = \frac{1}{2}\vec{J}^2$ with $J_k = -iX_k, \vec{J}^2 = j(j+1)\mathbf{1}, j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$
- $so(2, 1) \simeq su(1, 1)$:
 $g_{kl} = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \frac{1}{2}(-X_1^2 + X_2^2 + X_3^2) \quad \text{unbounded operator}$

Comment: The UIR of non-compact groups/algebras are ALL infinite dimensional. This becomes a problem for Lorentz group $SO(3, 1) = SO(3) \otimes SO(2, 1)$ and the classification of elementary particles. Way out is to use non-unitary but finite-dimensional irreducible reps. More later (Wigner states).

Generalisation of Racah:

$$C_n := c_{k_1 l_1}^{l_2} c_{k_2 l_2}^{l_3} \cdots c_{k_n l_n}^{l_1} X^{k_1} X^{k_2} \cdots X^{k_n}$$

commute with all X_i : $[C_n, X_i] = 0$

Range of $\mathcal{L} : \Leftrightarrow$ No. of independent C_n 's

4.5 Representations of Lie Algebras in Quantum Mechanics

4.5.1 The angular momentum algebra $so(3) \simeq su(2)$

Algebra:

$$[J_i, J_k] = i\hbar \varepsilon_{ikl} J_l$$

UIR have dimension $d_j = 2j + 1$ with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

$$\mathcal{D}^j = \text{span} \{|j, m\rangle | m = -j, \dots, j\} \simeq \mathbb{C}^{2j+1}$$

Cartan-Weyl basis: $J_{\pm} := J_1 \pm J_2$, $J_0 := J_3$

$$[J_0, J_{\pm}] = \pm \hbar J_{\pm}, \quad [J_+, J_-] = 2\hbar J_0$$

Usual basis: $J_0 |j, m\rangle = m\hbar |j, m\rangle \Rightarrow J_{\pm}$ change eigenvalue by $\pm \hbar$

Ansatz:

$$J_{\pm} |j, m\rangle = N_{\pm} \hbar |j, m \pm 1\rangle$$

Casimir: $C = \frac{1}{2} \vec{J}^2 \sim \mathbf{1} \Rightarrow \vec{J}^2 |j, m\rangle = \lambda_j |j, m\rangle$ with $\lambda_j \geq 0$ as $\vec{J}^2 \geq 0$.

Calculation of N_{\pm} : $(J_{\pm})^{\dagger} = J_{\mp}$, $J_0^{\dagger} = J_0$

$$\begin{aligned} |N_{\pm}|^2 \hbar^2 &= ||J_{\pm} |j, m\rangle||^2 = \langle j, m | J_{\mp} J_{\pm} |j, m\rangle = \langle j, m | [\vec{J}^2 - J_0(J_0 \pm 1)] |j, m\rangle \\ &= [\lambda_j - m(m \pm 1)] \hbar^2 \geq 0 \end{aligned}$$

Hence m must be bounded $m_{\min} \leq m \leq m_{\max}$ with $J_+ |j, m_{\max}\rangle = 0$ and $J_- |j, m_{\min}\rangle = 0$
Consider

$$\vec{J}^2 |j, m_{\max}\rangle = J_- J_+ |j, m_{\max}\rangle + J_0(J_0 + 1) |j, m_{\max}\rangle = m_{\max}(m_{\max} + 1) |j, m_{\max}\rangle$$

$$\vec{J}^2 |j, m_{\min}\rangle = J_+ J_- |j, m_{\min}\rangle + J_0(J_0 - 1) |j, m_{\min}\rangle = m_{\min}(m_{\min} - 1) |j, m_{\min}\rangle$$

With $\lambda_j := j(j + 1)$ we find $m_{\max} = j = -m_{\min}$ and

$$N_{\pm} = \sqrt{(j \mp m)(j \pm m + 1)} e^{i\phi_{\pm}}$$

4.5.2 The $so(4)$ symmetry of the H-atom

Classical Kepler problem:

$$E = \frac{\vec{p}^2}{2m} - \frac{\alpha}{r}, \quad \alpha = GMm$$

with

$$\vec{\ell} = \vec{r} \times \vec{p} = mr^2 \vec{\omega} = \text{const.}, \quad \frac{d}{dt} \vec{e}_r = \vec{\omega} \times \vec{e}_r$$

and Newton equation

$$\vec{F} = \dot{\vec{p}} = -\frac{\alpha}{r^2} \vec{e}_r$$

Laplace-Runge-Lenz vector:

$$\vec{A} := \vec{p} \times \vec{\ell} - m\alpha \vec{e}_r = \text{const.}, \quad \text{and} \quad \vec{A}^2 = m^2 \alpha^2 + 2m\vec{\ell}^2 E$$

Proofs:

- $\dot{\vec{A}} = \dot{\vec{p}} \times \vec{\ell} - m\alpha \dot{\vec{e}}_r = -\frac{\alpha}{r^2} \vec{e}_r \times (mr^2 \vec{\omega}) - m\alpha \vec{\omega} \times \vec{e}_r = \vec{0}$
- $\vec{A}^2 = (\vec{p} \times \vec{\ell})^2 - 2m\alpha \vec{e}_r \cdot (\vec{p} \times \vec{\ell}) + m^2 \alpha^2 = \vec{p}^2 \vec{\ell}^2 - (\vec{p} \cdot \vec{\ell})^2 - \frac{2m\alpha}{r} \vec{\ell} \cdot (\vec{r} \times \vec{p}) + m^2 \alpha^2 = \vec{p}^2 \vec{\ell}^2 - \frac{2m\alpha}{r} \vec{\ell}^2 + m^2 \alpha^2 = 2mE\vec{\ell}^2 + m^2 \alpha^2$

Quantum mechanical hydrogen atom:

Hamiltonian:

$$H = \frac{1}{2m} \vec{P}^2 - \frac{\alpha}{|\vec{Q}|} \quad \text{on} \quad \mathcal{H} = L^2(\mathbb{R}^3)$$

Angular Momentum:

$$\vec{L} = \vec{Q} \times \vec{P}$$

Laplace-Runge-Lenz vector: (re-scaled and symmetrized)

$$\vec{A} := \frac{1}{2m\alpha} \left(\vec{P} \times \vec{L} - \vec{L} \times \vec{P} \right) - \frac{\vec{Q}}{|\vec{Q}|}$$

\Rightarrow Two conserved vector operators

$$[H, \vec{L}] = \vec{0}, \quad [H, \vec{A}] = \vec{0}$$

and

$$\vec{A}^2 = 1 + \frac{2}{m\alpha^2} \left(\hbar^2 + \vec{L}^2 \right) H, \quad \vec{L} \cdot \vec{A} = 0 = \vec{A} \cdot \vec{L}$$

Algebra:

$$\begin{aligned} [A_i, A_j] &= i\hbar \varepsilon_{ijk} L_k \left(-\frac{2H}{m\alpha^2} \right) \\ [L_i, L_j] &= i\hbar \varepsilon_{ijk} L_k \\ [A_i, L_j] &= i\hbar \varepsilon_{ijk} A_k = [L_i, A_j] \end{aligned}$$

Consider subspace with fixed energy $E < 0$:

$$\mathcal{H}_E \subset L^2(\mathbb{R}^3) \quad \text{with} \quad \mathcal{H}_E := \{|\psi\rangle \in L^2(\mathbb{R}^3) | H|\psi\rangle = E|\psi\rangle\}$$

Redefine: $\vec{N} := \left(-\frac{2E}{m\alpha^2}\right)^{-1/2} \vec{A}$, $\vec{M} := \vec{L}$ then

$$\left. \begin{aligned} [M_i, M_j] &= i\hbar \varepsilon_{ijk} M_k \\ [M_i, N_j] &= i\hbar \varepsilon_{ijk} N_k \\ [N_i, N_j] &= i\hbar \varepsilon_{ijk} M_k \end{aligned} \right\} \quad so(4)\text{-algebra} \quad (\text{see homework})$$

Decoupling: $\vec{J} := \frac{1}{2}(\vec{M} + \vec{N})$ and $\vec{K} := \frac{1}{2}(\vec{M} - \vec{N})$

$$\left. \begin{aligned} [J_i, J_j] &= i\hbar \varepsilon_{ijk} J_k \\ [K_i, K_j] &= i\hbar \varepsilon_{ijk} K_k \\ [K_i, J_j] &= 0 \end{aligned} \right\} \quad so(4) \simeq so(3) \oplus so(3)$$

Consider UIR:

$$\begin{aligned} \vec{K}^2 |k, m_k\rangle &= k(k+1)\hbar^2 |k, m_k\rangle \\ \vec{J}^2 |j, m_j\rangle &= j(j+1)\hbar^2 |j, m_j\rangle \end{aligned}$$

Recall: $\vec{L} \cdot \vec{A} = 0 = \vec{M} \cdot \vec{K} = \vec{J}^2 - \vec{K}^2 \Rightarrow k = j$
Hence: $\mathcal{H}_E = D^k \otimes D^k$ product space of two UIR of $so(3)$
Product basis: $|k, m_k, m_j\rangle := |k, m_k\rangle \otimes |j, m_j\rangle$
Consider:

$$(\vec{J}^2 + \vec{K}^2)|k, m_k, m_j\rangle = 2k(k+1)\hbar^2|k, m_k, m_j\rangle$$

On the other hand:

$$\begin{aligned}\vec{J}^2 + \vec{K}^2 &= \frac{1}{2}(\vec{M}^2 + \vec{N}^2) \\ &= \frac{1}{2}\left(\vec{L}^2 - \frac{m\alpha^2}{2E}\vec{A}^2\right) \\ &= \frac{1}{2}\left(\vec{L}^2 - \frac{m\alpha^2}{2E}\left(1 + \frac{2E}{m\alpha^2}(\hbar^2 + \vec{L}^2)\right)\right) \\ &= -\frac{m\alpha^2}{4E} - \frac{\hbar^2}{2}\end{aligned}$$

$$\Rightarrow -\frac{m\alpha^2}{4E} = \frac{\hbar^2}{2} + 2k(k+1)\hbar^2 = \frac{\hbar^2}{2}(4k^2 + 4k + 1) = \frac{\hbar^2}{2}(2k+1)^2$$

$$\Rightarrow E = -\frac{m\alpha^2}{2\hbar^2} \frac{1}{(2k+1)^2} = -\frac{m\alpha^2}{2\hbar^2} \frac{1}{n^2} \quad \text{with} \quad n := 2k+1$$

Note: $k \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\} \Rightarrow n \in \{1, 2, 3, 4, \dots\}$

Degeneracy: $\dim \mathcal{H}_E = \dim(D^k \otimes D^k) = (2k+1)^2 \Rightarrow E = E_n$ is n^2 degenerate.

Angular momentum: $\vec{L} = \vec{J} + \vec{K}$ coupling of j and k

$$\begin{aligned}\Rightarrow \ell &\in \{|j-k|, |j-k|+1, \dots, j+k\} \quad \text{but} \quad k=j \\ &= \{0, 1, 2, \dots, 2k\} \quad \text{only integer} \quad \ell\end{aligned}$$

With $n = \ell + 1 + n_r$ and $m = m_k + m_j$ we change to new

$$|k, m_k, m_j\rangle \longrightarrow |n, \ell, m\rangle$$

Eigenstates with

$$\begin{aligned}H|n, \ell, m\rangle &= E_n|n, \ell, m\rangle \\ \vec{L}^2|n, \ell, m\rangle &= \hbar^2\ell(\ell+1)|n, \ell, m\rangle \\ L_z|n, \ell, m\rangle &= m\hbar|n, \ell, m\rangle\end{aligned}$$

Comments:

- For $E > 0$ one obtains a $SO(3, 1) = SO(3) \otimes SO(2, 1)$ symmetry. See later for an algebraic approach via $so(2, 1) \simeq su(1, 1)$ (spectrum-generating algebra).

In essence: $\vec{N} := \left(\frac{2E}{m\alpha^2}\right)^{-1/2} \vec{A}$

$$\left. \begin{aligned}[M_i, M_j] &= i\hbar\varepsilon_{ijk}M_k \\ [M_i, N_j] &= i\hbar\varepsilon_{ijk}N_k \\ [N_i, N_j] &= -i\hbar\varepsilon_{ijk}M_k\end{aligned} \right\} \quad so(3, 1)\text{-algebra}$$

Bertrand's Theorem: There are only two types of central-force (radial) scalar potentials with the property that all bound orbits are also closed orbits.

- The 3-D Kepler problem may also be mapped onto a 4-D harmonic oscillator problem via the so-called Kustaanheimo-Stiefel transformation (c.f. Homework Problem 8). This Newton-Hook duality was already known to both in 17th century.
- A fixed $SO(4)$ -UIR spans the subspace \mathcal{H}_E corresponding to a single energy shell of a bound state (fixed $n = 2k+1$ and varying ℓ, m).

- An irreducible representation of $SO(4, 1)$ spans the full bound spectrum. The group $SO(4, 1)$ is also called de Sitter group (Willem de Sitter 1872–1934). de Sitter space is a maximally symmetric Lorentzian manifold with constant positive scalar curvature. Embed S^3 in \mathbb{R}^4
- An $SO(3, 2)$ irreducible representation spans the full continuous spectrum. The group $SO(3, 2)$ is also called anti-de Sitter group. Anti-de Sitter space is a maximally symmetric Lorentzian manifold with constant negative scalar curvature.
- $SO(4, 2)$ is called the full dynamical group of the Kepler (or Hydrogen atom problem). It is the smallest group whose irreducible representations span both the continuous and the discrete spectrum.

Some proofs:

$$\vec{L} \cdot \vec{A} = 0 = \vec{A} \cdot \vec{L} \text{ is obvious as } \vec{L} \cdot (\vec{P} \times \vec{L}) = 0 \text{ and } \vec{L} \cdot \vec{Q} = 0$$

$$\text{With } \vec{L} \times \vec{P} = 2i\hbar\vec{P} - \vec{P} \times \vec{L} \text{ and } R := |\vec{Q}| \text{ follows}$$

$$\vec{A} = \frac{1}{\alpha m} \left(\vec{P} \times \vec{L} - i\hbar\vec{P} \right) - \frac{\vec{Q}}{R}$$

\Rightarrow

$$\alpha^2 m^2 (\vec{A}^2 - 1) = (\vec{P} \times \vec{L} - i\hbar\vec{P})^2 - \alpha m (\vec{P} \times \vec{L} - i\hbar\vec{P}) \cdot \frac{\vec{Q}}{R} - \alpha m \frac{\vec{Q}}{R} \cdot (\vec{P} \times \vec{L} - i\hbar\vec{P})$$

Using following relations (proofs are below)

$$\begin{aligned} (\vec{P} \times \vec{L}) \cdot (\vec{P} \times \vec{L}) &= \vec{P}^2 \vec{L}^2 \\ (\vec{P} \times \vec{L}) \cdot \vec{P} &= 2i\hbar\vec{P}^2 \\ \vec{P} \cdot (\vec{P} \times \vec{L}) &= 0 \\ (\vec{P} \times \vec{L}) \cdot \vec{Q} &= \vec{L}^2 + 2i\hbar\vec{P} \cdot \vec{Q} \\ \vec{Q} \cdot (\vec{P} \times \vec{L}) &= \vec{L}^2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} (\vec{P} \times \vec{L} - i\hbar\vec{P})^2 &= \vec{P}^2 (\vec{L}^2 + \hbar^2) \\ (\vec{P} \times \vec{L} - i\hbar\vec{P}) \cdot \vec{Q} &= \vec{L}^2 + i\hbar\vec{P} \cdot \vec{Q} \\ \vec{Q} \cdot (\vec{P} \times \vec{L} - i\hbar\vec{P}) &= \vec{L}^2 - i\hbar\vec{Q} \cdot \vec{P} \end{aligned}$$

follows

$$\alpha^2 m^2 (\vec{A}^2 - 1) = \vec{P}^2 (\vec{L}^2 + \hbar^2) - 2\alpha m \frac{\vec{L}^2}{R} - \underbrace{i\hbar\alpha m \left(\vec{P} \cdot \frac{\vec{Q}}{R} - \frac{\vec{Q}}{R} \cdot \vec{P} \right)}_{-2i\hbar/R} = 2mH (\vec{L}^2 + \hbar^2)$$

Auxiliary formulas:

- $(\vec{P} \times \vec{L}) \cdot (\vec{P} \times \vec{L}) = \varepsilon_{ijk} P_j L_k \varepsilon_{ilm} P_l L_m = \varepsilon_{ijk} \varepsilon_{ilm} P_j L_k P_l L_m = \varepsilon_{ijk} \varepsilon_{ilm} P_j (P_l L_k + i\hbar \varepsilon_{klr} P_r) L_m = (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) (P_j P_l L_k L_m + i\hbar \varepsilon_{klr} P_j P_r L_m) = \vec{P}^2 \vec{L}^2 + i\hbar \varepsilon_{kjr} P_j P_r L_k - P_j \vec{P} \cdot \vec{L} L_j - i\hbar \varepsilon_{kkr} P_j P_r L_j = \vec{P}^2 \vec{L}^2$
- $(\vec{P} \times \vec{L}) \cdot \vec{P} = \varepsilon_{ijk} P_i L_j P_k = \varepsilon_{ijk} P_i (P_k L_j + i\hbar \varepsilon_{jkl} P_l) = i\hbar 2\delta_{il} P_i P_l = 2i\hbar \vec{P}^2$
- $\vec{P} \cdot (\vec{P} \times \vec{L}) = \varepsilon_{ijk} P_i P_j L_k = 0$
- $(\vec{P} \times \vec{L}) \cdot \vec{Q} = \varepsilon_{ijk} P_i L_j Q_k = \varepsilon_{ijk} P_i (Q_k L_j + i\hbar \varepsilon_{jkl} Q_l) = \varepsilon_{ijk} Q_k P_i L_j + i\hbar 2\delta_{il} P_i Q_l = \vec{L}^2 + 2i\hbar \vec{P} \cdot \vec{Q}$
- $\vec{Q} \cdot (\vec{P} \times \vec{L}) = \varepsilon_{ijk} Q_i P_j L_k = \vec{L}^2$
- $(\vec{P} \times \vec{L} - i\hbar\vec{P})^2 = (\vec{P} \times \vec{L}) \cdot (\vec{P} \times \vec{L}) - i\hbar(\vec{P} \times \vec{L}) \cdot \vec{P} - i\hbar\vec{P} \cdot (\vec{P} \times \vec{L}) - \hbar^2 \vec{P}^2 = \vec{P}^2 \vec{L}^2 - 2i\hbar \vec{P}^2 - \hbar^2 \vec{P}^2 = \vec{P}^2 (\vec{L}^2 + \hbar^2)$
- $(\vec{P} \times \vec{L} - i\hbar\vec{P}) \cdot \vec{Q} = \vec{L}^2 + 2i\hbar \vec{P} \cdot \vec{Q} - i\hbar\vec{P} \cdot \vec{Q} = \vec{L}^2 + i\hbar \vec{P} \cdot \vec{Q}$
- $\vec{Q} \cdot (\vec{P} \times \vec{L} - i\hbar\vec{P}) = \vec{L}^2 - i\hbar \vec{Q} \cdot \vec{P}$
- $\vec{P} \cdot \frac{\vec{Q}}{R} - \frac{\vec{Q}}{R} \cdot \vec{P} = \frac{\hbar}{1} (\vec{\nabla} \cdot \vec{Q}) \frac{1}{R} + \frac{\hbar}{1} \vec{Q} \cdot (\vec{\nabla} \frac{1}{R}) = \frac{\hbar}{1} \frac{3}{R} - \frac{\hbar}{1} \vec{Q} \cdot \frac{\vec{Q}}{R^3} = \frac{\hbar}{1} \frac{2}{R}$

Finally the proof for $[\vec{A}, H] = 0$

With

$$\vec{P} \times \vec{L} = P^2 \vec{Q} - (\vec{P} \cdot \vec{Q}) \vec{P} + i\hbar \vec{P} \quad \text{and} \quad \vec{L} \times \vec{P} = -P^2 \vec{Q} + (\vec{P} \cdot \vec{Q}) \vec{P} + i\hbar \vec{P}$$

follows

$$\vec{A} = \frac{1}{m\alpha} \left(P^2 \vec{Q} - (\vec{P} \cdot \vec{Q}) \vec{P} \right) - \frac{\vec{Q}}{R}$$

Using following formulas

- $[\vec{P}, 1/R] = i\hbar \frac{\vec{Q}}{R^3}$
- $[P^2, 1/R] = i\hbar \frac{1}{R^3} (\vec{Q} \cdot \vec{P}) + i\hbar (\vec{P} \cdot \vec{Q}) \frac{1}{R^3}$
- $[(\vec{P} \cdot \vec{Q}) \vec{P}, 1/R] = i\hbar (\vec{P} \cdot \vec{Q}) \frac{1}{R^3} + i\hbar \frac{1}{R} \vec{P}$

one finds

$$\begin{aligned} m\alpha [\vec{A}, \frac{1}{R}] &= [P^2 \vec{Q}, \frac{1}{R}] - [(\vec{P} \cdot \vec{Q}) \vec{P}, \frac{1}{R}] \\ &= i\hbar \frac{1}{R^3} (\vec{Q} \cdot \vec{P}) \vec{Q} + i\hbar (\vec{P} \cdot \vec{Q}) \frac{\vec{Q}}{R^3} - i\hbar (\vec{P} \cdot \vec{Q}) \frac{\vec{Q}}{R^3} - i\hbar \frac{1}{R} \vec{P} \\ &= i\hbar \frac{1}{R^3} (\vec{Q} \cdot \vec{P}) \vec{Q} - i\hbar \frac{1}{R} \vec{P} \end{aligned}$$

Consider now

$$\begin{aligned} [\vec{A}, P^2] &= \frac{1}{\alpha m} \left([P^2 \vec{Q}, P^2] - [(\vec{P} \cdot \vec{Q}) \vec{P}, P^2] \right) - [\frac{\vec{Q}}{R}, P^2] \\ &= \frac{1}{\alpha m} \left(P^2 [\vec{Q}, P^2] - [(\vec{P} \cdot \vec{Q}), P^2] \vec{P} \right) - \frac{1}{R} [\vec{Q}, P^2] - [\frac{1}{R}, P^2] \vec{Q} \\ &= \frac{1}{\alpha m} \underbrace{\left(P^2 2i\hbar \vec{P} - \vec{P} \cdot [\vec{Q}, P^2] \vec{P} \right)}_{=0} - \frac{1}{R} [\vec{Q}, P^2] - [\frac{1}{R}, P^2] \vec{Q} \\ &= -\frac{1}{R} 2i\hbar \vec{P} + i\hbar \frac{1}{R^3} (\vec{Q} \cdot \vec{P}) \vec{Q} + i\hbar (\vec{P} \cdot \vec{Q}) \frac{\vec{Q}}{R^3} \quad \text{with} \quad [\vec{P} \cdot \vec{Q}, \frac{1}{R^3}] = 3i\hbar \frac{1}{R^3} \\ &= -2i\hbar \frac{1}{R} \vec{P} + i\hbar \frac{1}{R^3} \left((\vec{Q} \cdot \vec{P}) + \underbrace{(\vec{P} \cdot \vec{Q})}_{(\vec{Q} \cdot \vec{P})} + 3i\hbar \right) \vec{Q} \\ &= -2i\hbar \frac{1}{R} \vec{P} + 2i\hbar \frac{1}{R^3} (\vec{Q} \cdot \vec{P}) \vec{Q} \\ &= 2m\alpha [\vec{A}, \frac{1}{R}] \end{aligned}$$

Hence

$$\frac{1}{2m} [\vec{A}, P^2] = [\vec{A}, \alpha/R] \quad \Rightarrow \quad [\vec{A}, H] = 0$$

Good References:

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*** End of Lecture 4 ***