Recall orthogonality relation for matrix elements of UIR:

$$\boxed{\frac{1}{n}\sum_{g\in G} D^i_{\mu\nu}(g) D^{j\,*}_{\rho\sigma}(g) = \frac{1}{d_i}\delta_{ij}\delta_{\mu\rho}\delta_{\nu\sigma}}$$

Recall Peter-Weyl-Theorem: $f: G \to \mathbb{C}$

$$\boxed{\begin{aligned} f(g) &= \sum_{\text{all UIR } i} d_i \sum_{\mu,\nu=1}^{d_i} \tilde{f}^i_{\nu\mu}\, D^i_{\mu\nu}(g) \\ \tilde{f}^i_{\nu\mu} &= \frac{1}{n}\sum_{g\in G} f(g)\left(D^i_{\mu\nu}(g)\right)^* \end{aligned}}$$

For Abelian groups all UIR are 1-dimensional, i.e. D^i is a complex number and $d_i = 1$. Peter-Weyl-Theorem for Abelian groups:

$$\boxed{\begin{aligned} f(g) &= \sum_{\text{all UIR } i} \tilde{f}^{\,i}\, D^i(g) \\ \tilde{f}^{\,i} &= \frac{1}{n}\sum_{g\in G} f(g)\left(D^i(g)\right)^* \end{aligned}}$$

Example: $Z_2 = \{-1, +1\}$, or $g = \sigma = \pm 1$. 2 classes $\Rightarrow$ 2 UIR: $D^0(g) = 1$ and $D^1(g) = \sigma$. Let $f(g) := f_\sigma$ with $f_\sigma \in \mathbb{C}$.

$$f(g) = \tilde{f}^0 + \tilde{f}^1\sigma \qquad \text{where} \qquad \tilde{f}^0 = \frac{1}{2}(f_+ + f_-)\,, \qquad \tilde{f}^1 = \frac{1}{2}(f_+ - f_-)$$

Check: $f(g) = \frac{1}{2}(f_+ + f_-) + \frac{\sigma}{2}(f_+ - f_-)$

2.7.3 Characters of representations

Definition: The function

$$\chi^j : \begin{array}{l} G \to \mathbb{C} \\ g \mapsto \chi^j(g) := \operatorname{Tr} D^j(g) \end{array}$$

is called *Character* of representation D^j with finite dimension d_j.

Comments:

- Equivalent reps have the same character as $\operatorname{Tr} S^{-1}D(g)S = \operatorname{Tr} D(g)$.
- Characters are *class functions* (functions on classes of a group) as for g_1 and g_2 being within same class exists a $g \in G$ with $g_1 = g g_2 g^{-1}$. Hence $\chi^j(g_1) = \chi^j(g g_2 g^{-1}) = \chi^j(g_2)$, that is, is constant within the class.
- Characters of UIR are orthogonal

  $$\frac{1}{n}\sum_{g\in G}\chi^i(g)\chi^{j\,*}(g) = \delta_{ij}$$

  That is, $\{\chi^i\}$ is complete orthogonal set for class functions. Recall orthogonality relation above. (Proof as little Exercise)
- UIR are uniquely characterised by the characters. Consider fully reducible reps $D(g) = \bigoplus_j c_j\, D^j(g)$ with D^i UIR, then $\chi(g) = \sum_j c_j \chi^j(g)$ with

  $$c_j = \frac{1}{n}\sum_{g\in G}\chi(g)\chi^{j\,*}(g)$$

  The decomposition of D into UIR is unique!

Example: Let us consider the regular representation

$$g_\mu g_\nu =: \sum_{\rho=1}^{n} D^{\text{reg}}_{\rho\nu}(g_\mu) g_\rho$$

which is fully reducible and n-dimensional. Then, $\chi^{\text{reg}}(e) = n$ and $\chi^{\text{reg}}(g) = 0$ for all $g \neq e$. That is,

$$c_j = \frac{1}{n}\sum_g \chi^{\text{reg}}(g)\chi^{j\,*}(g) = \frac{1}{n} n \chi^{j\,*}(e) = d_j$$

for all UIR j. Hence, all UIR of a finite group have multiplicity d_j (their dimension) in the regular representation. That is,

$$\chi^{\text{reg}}(g) = \sum_{\text{all UIR}} d_j\, \chi^j(g)$$

In above let $g = e$, then

$$\chi^{\text{reg}}(e) = n = \sum_{\text{all UIR}} d_j\, \chi^j(e) = \sum_{\text{all UIR}} d_j^2$$

which proofs the theorem of Burnside.
The problem of finding all UIR is equivalent to the full reduction of the regular representation.

Theorem: The number of inequivalent UIR of a finite group is identical to the number of classes.

Proof: Consider arbitrary class function $f(g) = f(g_0^{-1} g g_0)$ for all $g, g_0 \in G$.
From Peter-Weyl theorem follows

$$f(g) = \sum_j d_j \sum_{\mu\nu} \tilde{f}^j_{\nu\mu}\, D^j_{\mu\nu}(g_0^{-1} g g_0)$$

Now take group average over g_0

$$\begin{aligned}
f(g) &= \sum_j d_j \sum_{\mu\nu} \tilde{f}^j_{\nu\mu} \frac{1}{n} \sum_{g_0} \sum_{\alpha\beta} D^j_{\mu\alpha}(g_0^{-1}) D^j_{\alpha\beta}(g) D^j_{\beta\nu}(g_0) \\
&\quad \text{use} \qquad \frac{1}{n}\sum_{g_0} D^j_{\mu\alpha}(g_0^{-1}) D^j_{\beta\nu}(g_0) = \delta_{\mu\nu}\delta_{\alpha\beta} \\
&= \sum_j d_j \sum_{\mu\nu} \tilde{f}^j_{\nu\mu} \sum_{\alpha\beta} \frac{1}{d_j} \delta_{\mu\nu}\delta_{\alpha\beta} D^j_{\alpha\beta}(g) \\
&= \sum_j \sum_\mu \tilde{f}^j_{\mu\mu} \sum_\alpha D^j_{\alpha\alpha}(g) \\
&= \sum_j \operatorname{Tr}(\tilde{f}^j)\, \chi^j(g)
\end{aligned}$$

with $\operatorname{Tr}(\tilde{f}^j) := \sum_\mu \tilde{f}^j_{\mu\mu} = \frac{1}{n}\sum_{g\in G} f(g)\chi^j(g^{-1})$

Peter-Weyl theorem for class functions $f(g) = f(g_0^{-1} g g_0)$:

$$\boxed{\begin{aligned}
f(g) &= \sum_{\text{all UIR } i} a_i\, \chi^i(g) \\
a_i &= \frac{1}{n}\sum_{g\in G} f(g)\chi^{i\,*}(g)
\end{aligned}}$$

$\Rightarrow \{\chi^j(g)\}$ is a complete orthonormal set for class functions, which are constant on a class
$\Rightarrow \vec{f} = (f(g_1), f(g_2), \ldots, f(g_k))$, $g_i \in K_i$ (i-th class), $i = 1, 2, \ldots, k = \#$ of classes of G,
$\vec{f}$ is an element of a k-dimensional vector space $\mathcal{K} \subset \mathbb{C}^n$, $k \leq n$
$\Rightarrow$ there exist exact k UIR for a finite group.

Character tables: Are used for finite groups of low order to classify their UIR

G	$g_1 = e$	$g_2 \in K_2$	$\cdots$	$g_k \in K_k$	
D^0	1	1	$\cdots$	1	trivial reps
$\vdots$					
D^i	d_i	$\chi^i(g_2)$		$\chi^i(g_k)$	i-th reps

Construction:

- # of UIR = # classes $\Rightarrow$ quadratic table
- Burnside: $\sum_i d_i^2 = n = \operatorname{ord} G$
- $\sum_{g \in G} \chi^i(g) \chi^{j\,*}(g) = \sum_{\ell=1}^{k} m_\ell \, \chi^i(g_\ell) \chi^{j\,*}(g_\ell) = n \delta_{ij}$,
  where $g_\ell \in K_\ell$ and $m_\ell = \#$ of elements in class K_ℓ.
  $\Rightarrow$ sum rule for each row i
  $$\sum_{\ell=1}^{k} m_\ell \, |\chi^i(g_\ell)|^2 = n$$

Example: $C_3 = \{e, d, d^2\}$ has 3 classes, abelian, $n = 3$ and $d^3 = e \Rightarrow \left[\chi^i(d)\right]^3 = 1$
$\Rightarrow \chi^i(d) \in \left\{1, \mathrm{e}^{2\pi \mathrm{i}/3}, \mathrm{e}^{4\pi \mathrm{i}/3}\right\}$

C_3	e	d	d^2
D^0	1	1	1
D^1	1	$\mathrm{e}^{2\pi \mathrm{i}/3}$	$\mathrm{e}^{4\pi \mathrm{i}/3}$
D^2	1	$\mathrm{e}^{4\pi \mathrm{i}/3}$	$\mathrm{e}^{2\pi \mathrm{i}/3}$

Projection Operators: Let G be a finite group, D a unitary fully reducible reps in some vector space V, χ^j the character of the UIR labeled with j and dimension d_j.

Theorem: The operator
$$\mathbb{E}^j := \frac{d_j}{n} \sum_{g \in G} \chi^{j\,*}(g) D(g)$$
fulfills following relations

1. $\mathbb{E}^{j\,\dagger} = \mathbb{E}^j$ self-adjoint
2. $\mathbb{E}^j \mathbb{E}^k = \mathbb{E}^j \delta_{jk}$ ortho-normal projector
3. $\sum_j \mathbb{E}^j = \mathbf{1}$ completeness
4. $D(g)\mathbb{E}^j = \mathbb{E}^j D(g)$

Proof: See Homework Problem 7

Comments:

- $\mathbb{E}^j$ is ortho-normal projector onto invariant subspace of V spanned by the UIR j within D (with possible multiplicities)

- $\mathbb{E}^j$ can be used to find invariant subspaces
- Extension to compact groups obvious

$$\mathbb{E}^j = d_j \int_G \mathrm{d}g \, \chi^{j\,*}(g) D(g)$$

- $\mathbb{E}^0 = \frac{1}{n} \sum_g D(g)$ or $\mathbb{E}^0 = \int_G \mathrm{d}g \, D(g)$
  projects on invariant subspace of trivial reps = average of D on group

3 Lie Groups

3.1 Pragmatic Approach to Lie Groups

For some more details please see, e.g., book by Lucha & Schöberl.
A *continuous* or *topological* group has uncountable infinite group elements.

Parametrisation and Notation:

- $g = g(\alpha) = g(\alpha_1, \alpha_2, \ldots, \alpha_n)$,
- $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ independent group parameters, that is, $g(\alpha) \neq g(\alpha') \Leftrightarrow \alpha \neq \alpha'$
- $\alpha \in I \subseteq \mathbb{R}^n$, I is *group space*, one or more not necessarily connected subsets of $\mathbb{R}^n$
- $n \in \mathbb{N}$ is the *dimension* of the group
- Convention for neutral element $e = g(0) = g(0, 0, \ldots, 0)$

Examples:

- $SO(2)$: $g = g(\varphi)$, $\varphi \in [0, 2\pi[\subset \mathbb{R}$, $I = S^1$ unit circle, 1-dim. continuous group
- T^3: $g = g(\vec{x})$, $\vec{x} \in \mathbb{R}^3$, $I = \mathbb{R}^3$, 3-dim. continuous group

Composition laws and composition functions:

$$\text{multiplication} \qquad g(\gamma) = g(\alpha)g(\beta) \qquad \Rightarrow \exists\, \Phi : \begin{array}{l} I \times I \to I \\ (\alpha, \beta) \mapsto \gamma = \Phi(\alpha, \beta) \end{array}$$

$$\text{inverse element} \qquad g(\alpha') = g^{-1}(\alpha) \qquad \Rightarrow \exists\, \Psi : \begin{array}{l} I \to I \\ \alpha \mapsto \alpha' = \Psi(\alpha) \end{array}$$

Properties:

$$\begin{array}{lcl} g(\gamma)(g(\beta)g(\alpha)) = (g(\gamma)g(\beta))g(\alpha) & \Rightarrow & \Phi(\gamma, \Phi(\beta, \alpha)) = \Phi(\Phi(\gamma, \beta), \alpha) \\ g(0)g(\alpha) = g(\alpha)g(0) & \Rightarrow & \Phi(0, \alpha) = \alpha = \Phi(\alpha, 0) \\ g(\alpha)g^{-1}(\alpha) = g^{-1}(\alpha)g(\alpha) & \Rightarrow & \Phi(\alpha, \Psi(\alpha)) = 0 = \Phi(\Psi(\alpha), \alpha) \end{array}$$

These are rather strong conditions!

Definition: *Topological* or *Continuous* group

- I is *topological* space not necessarily connected (limits, continuity, connectedness)
- Composition functions are *continuous*

Definition: *Lie Group* is topological group with

- I is *analytical manifold* not necessarily connected (manifold with analytic atlas = analytic transformation functions)
- Composition functions are *analytic*

Examples:

- $SO(2)$:

$$g(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

  Composition laws = trigonometric addition theorems

- $O(2)$:

$$g_d(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \qquad \text{pure rotations} \qquad \det g_d(\varphi) = 1$$

$$g_s(\varphi) = g_d(\varphi) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \qquad \text{rot. \& refl.} \qquad \det g_d(\varphi) = -1$$

  Group space not connected , $SO(2)$ is normal subgroup $O(2)/SO(2) \simeq Z_2$

Comment: Let $G_0 \subset G$ be the connected subset containing $e = g(0) \Rightarrow G_0$ is normal subgroup of G.

Compact Groups: A topological group is called compact when its group space I is compact. The group space may consist of several compact components.

- $SO(2)$: $I = S^1$ unit circle is compact
- $SO(1,1)$: Boosts in $(1+1)$ dimensions

$$g(\beta) := \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} . \qquad \beta \in \mathbb{R}$$

  $I \simeq \mathbb{R}$ hyperbola unbounded, is NOT compact but locally compact

Locally compact Groups: If for all $g \in G$ there exists a compact environment (incl. boundaries) which is completely within G, then G is called *locally compact*.

3.2 Invariant Measure for Topological Groups

Basic Assumption:
There exists a positive measure μ on G, that is, for any μ-measureable function

$$f: \begin{array}{l} G \to \mathbb{C} \\ g \mapsto f(g) \end{array}$$

the integration over a topological group is well-defined:

$$\int_G \mathrm{d}\mu(g)\, f(g) = \int_I \mathrm{d}^n\alpha\, \rho(\alpha) f(g(\alpha))$$

$\mathrm{d}^n\alpha$: usual Lebesque measure
$\rho(\alpha)$: density of group elements at α

Definitions: For all μ-measurable f and all $g_0 \in G$

- *Left-invariant Haar measure*:

$$\int_G \mathrm{d}\mu(g)\, f(g_0 g) = \int_G \mathrm{d}\mu(g)\, f(g) \qquad \Leftrightarrow \qquad \mu(g_0 g) = \mu(g)$$

- *Right-invariant Haar measure*:

$$\int_G \mathrm{d}\mu(g)\, f(g g_0) = \int_G \mathrm{d}\mu(g)\, f(g) \qquad \Leftrightarrow \qquad \mu(g g_0) = \mu(g)$$

- *Invariant Haar measure*:

$$\int_G \mathrm{d}\mu(g)\, f(g_0 g g_1) = \int_G \mathrm{d}\mu(g)\, f(g) \qquad \Leftrightarrow \qquad \mu(g_0 g g_1) = \mu(g)$$

Example: $G = SO(2)$, $g = g(\varphi)$.
Let $g_0 = g(\alpha)$ and $g_1 = g(\beta)$ then $g_0 g g_1 = g(\alpha + \varphi + \beta)$ and
$f(g) = f(g(\varphi)) = f(g(\varphi + 2\pi))$ periodic function on unit circle
$\Rightarrow \mathrm{d}\mu(g) = \frac{1}{2\pi}\mathrm{d}\varphi$ is invariant Haar measure for $SO(2)$ as

$$\int_{SO(2)} \mathrm{d}\mu(g)\, f(g_0 g g_1) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\, f(g(\alpha+\varphi+\beta)) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\tilde{\varphi}\, f(g(\tilde{\varphi})) = \int_{SO(2)} \mathrm{d}\mu(g)\, f(g)$$

Theorem:
For each *locally compact group* there exists a (non-trivial) positive left-invariant measure which is, up to a (normalization) constant, unique.

Proof: See for example,
J. Dieudomé, *Grundzüge der modernen Analysis II*, Chapter 14.1, pp249-255.

Normalization:

- For compact groups: $\displaystyle\int_G \mathrm{d}\mu(g) = 1 = \mu(G)$
- For finite groups: $\displaystyle\frac{1}{\mathrm{ord}\, G}\sum_{g\in G} 1 = 1$
- For infinite discrete groups: $\mu(e) = 1$

Comments:

- Construction of a left-invariant measure for Lie groups always possible in principle (see E. Wigner, *Group Theory*, p. 95-99 and optional tutorial after test). In practice this might be difficult for non-abelian groups
- The existence is often sufficient even without explicitly knowing the density ρ.
- An educated guess of the measure is usually faster than its formal construction a la Wigner

Modular function of a locally compact group G
Let μ be the left-invariant measure on G, i.e. $\mu(g_0 g) = \mu(g)$.
Then obviously $\mu(g_0 g g_1) = \mu(g g_1)$ is also left-invariant.
Hence, uniqueness implies that $\mu(g g_1) = \Delta_G(g_1)\mu(g)$, where

$$\Delta_G : \begin{array}{l} G \to \mathbb{R}^+ \\ g \mapsto \Delta_G(g) \end{array}$$

is called the *modular* function of G.

Definition:

$$G \text{ uni-modular} \qquad :\Leftrightarrow \qquad \Delta_G(g) = 1$$

$\Rightarrow \qquad \mu(g g_1) = \mu(g)$ is also right-invariant $\qquad \Rightarrow \qquad$ invariant Haar measure

Notation: For uni-modular groups $\mathrm{d}\mu(g) = \mathrm{d}g$ from now on
and

$$\boxed{\int_G \mathrm{d}g\, f(g) = \int_G \mathrm{d}g\, f(g_0 g) = \int_G \mathrm{d}g\, f(g g_0) = \int_G \mathrm{d}g\, f(g^{-1})}$$

Example: $SU(2)$

$$g = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1, \qquad a, b \in \mathbb{C}$$

Choose parametrization with Euler angles (bi-polar coordinates on $S^3 \subset \mathbb{R}^4$):

$$a = \cos\frac{\theta}{2} \exp\left\{\mathrm{i}\frac{\varphi+\psi}{2}\right\} \qquad 0 \le \varphi < 2\pi$$
$$0 \le \theta \le \pi$$
$$b = \mathrm{i}\sin\frac{\theta}{2} \exp\left\{\mathrm{i}\frac{\varphi-\psi}{2}\right\} \qquad -2\pi \le \psi < 2\pi$$

Then

$$\mathrm{d}g = \frac{1}{16\pi^2} \sin\theta \mathrm{d}\theta\, \mathrm{d}\varphi\, \mathrm{d}\psi$$

Proof: Consider bi-polar coordinates in $\mathbb{R}^4$

$$\begin{aligned} x_1 &= r\cos\tfrac{\theta}{2}\cos\tfrac{\varphi+\psi}{2} \\ x_2 &= r\cos\tfrac{\theta}{2}\sin\tfrac{\varphi+\psi}{2} \\ x_3 &= r\sin\tfrac{\theta}{2}\cos\tfrac{\varphi-\psi}{2} \\ x_2 &= r\sin\tfrac{\theta}{2}\sin\tfrac{\varphi-\psi}{2} \end{aligned} \qquad \mathrm{d}^4x = \underbrace{\left|\frac{\partial(x_1,x_2,x_3,x_4))}{\partial(r,\theta,\varphi,\psi)}\right|}_{=\frac{r^3}{8}\sin\theta} \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}\psi$$

Hence $\mathrm{d}^4x = \mathrm{d}^3\Omega\, r^3\mathrm{d}r$ with $\mathrm{d}^3\Omega = \frac{1}{8}\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}\psi$
Obviously d^4x is invariant under $SU(2)$ rotations in $\mathbb{R}^4$ leaving r fixed.
Hence, $\mathrm{d}^3\Omega$ is also $SU(2)$ invariant measure on $SU(2) \simeq S^3$.
Noting that $\int_{S^3} \mathrm{d}^3\Omega = 2\pi^2$ provides us with above normalised Haar measure

List of some uni-modular groups:

- All discrete groups
- All compact groups
- All Abelian groups
- $GL(n,\mathbb{R}) = \{X|\text{real } n \times n \text{ matrices with } \det x \neq 0\}$
- ...

Left-invariant measure of $GL(n,\mathbb{R})$:

$$g_X = X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix} \qquad \Rightarrow \qquad \mathrm{d}g_X = |\det X|^{-n} \prod_{i,j=1}^{n} \mathrm{d}x_{ij}$$

Left-invariance: Let $Y = AX$ that is $y_{ij} = \sum_{k=1}^{n} a_{ik}x_{kj}$

$$\Rightarrow \frac{\partial(y_{11},\ldots,y_{nn})}{\partial(x_{11},\ldots,x_{nn})} = (\det A)^n$$

Hence

$$\begin{aligned} \mathrm{d}g_Y &= |\det Y|^{-n} \prod_{i,j=1}^{n} \mathrm{d}y_{ij} = |\det A \cdot \det X|^{-n} \left|\frac{\partial(y_{11},\ldots,y_{nn})}{\partial(x_{11},\ldots,x_{nn})}\right| \prod_{i,j=1}^{n} \mathrm{d}x_{ij} \\ &= |\det X|^{-n} \prod_{i,j=1}^{n} \mathrm{d}x_{ij} = \mathrm{d}g_X = \mathrm{d}g_{AX} \end{aligned}$$

Right-invariance analogous

Consider subgroup of triangular matrices

$$g_Z = Z = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ 0 & z_{22} & \cdots & z_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & z_{nn} \end{pmatrix} \qquad \text{is NOT uni-modular}$$

Left-invariant measure: $\mathrm{d}\mu_L(g_Z) = |z_{11}^n z_{22}^{n-1} \cdots z_{nn}|^{-1} \prod_{i \leq j} \mathrm{d}z_{ij}$

Right-invariant measure: $\mathrm{d}\mu_L(g_Z) = |z_{11} z_{22}^2 \cdots z_{nn}^n|^{-1} \prod_{i \leq j} \mathrm{d}z_{ij}$

Remark: Subgroups of uni-modular groups are not necessarily uni-modular!

3.3 Transformation Groups

Transformation: Bijective mapping of a set $\mathcal{M}$ onto $\mathcal{M}$

$$g: \begin{array}{c} \mathcal{M} \to \mathcal{M} \\ x \mapsto gx \end{array}$$

Transformation Group: Exists for all $g \in G$ a transformation on $\mathcal{M}$ such that $ex = x$ for all $x \in \mathcal{M}$ (identical transformation) and $(g_1 g_2)x = g_1(g_2)x$ for all $g_1, g_2 \in G$, then G is called transformation group acting on $\mathcal{M}$.

Examples:

- $\mathcal{M} = \{x_1, x_2, \dots, x_n\}$ $\quad G = S_n$ Permutations
- $\mathcal{M} = \mathbb{R}^3$ $\quad G = SO(3)$ or $G = T^3$
- $\mathcal{M} = G$ obvious, for example $SU(2) \simeq S^3$

Transformation groups are called

effective $:\Leftrightarrow$ $\forall\, g \neq e\, \exists\, x \in \mathcal{M}$ such that $gx \neq x$

transitive $:\Leftrightarrow$ $\forall\, x, y \in \mathcal{M}\, \exists\, g \in G$ such that $gx = y$

Obviously $SO(3)$ is NOT transitive on $\mathcal{M} = \mathbb{R}^3$ but it is transitive on $\mathcal{M} = S^2$

Homogenous Space:
Exists a transitive group G acting on $\mathcal{M}$ then $\mathcal{M}$ is called *homogenous* space

From now on we will ONLY consider homogenous spaces $\mathcal{M}$ and transitive transformation groups G

Stationary Subgroup: Also called little group or isotropy group
Let $x_0 \in \mathcal{M}$ and G be transitive on $\mathcal{M}$, then

$$H := \{h \in G | hx_0 = x_0\}$$

is a subgroup of G called *stationary subgroup* of G with respect to $x_0 \in \mathcal{M}$.

Proof: Consider arbitrary $h_1, h_2 \in H$

- $h_1^{-1} h_1 x_0 = h_1^{-1} x_0 \Rightarrow h_1 x_0 = x_0 \Rightarrow h_1^{-1} \in H$
- $h_1 h_2 x_0 = h_1 x_0 = x_0 \Rightarrow h_1 h_2 \in H$

- $ex_0 = x_0 \Rightarrow e \in H$

$\Rightarrow H$ is a group

Example: Let $\mathcal{M} = S^2$, $G = SO(3)$ and choose $x_0 = \vec{e}_z$
$\Rightarrow H = SO(2)$ all rotation about z-axis keeping x_0 fixed (stationary)

In general, let $gx_0 = x$, that is for all $\tilde{g} = gh$ with $h \in H$ we have $\tilde{g}x_0 = x$.
That is, the set of all transformations mapping $x_0 \to x$ is represented by gH (left coset).
For each pair (x, x_0) exists a left coset gH such that $gHx_0 = x$
$\Rightarrow$ The homogenous space $\mathcal{M} \simeq$ set of all cosets

Notation: $\mathcal{M} = G/H := \{gH | g \in G\}$ for homogenous spaces
is in general NOT a (factor) group as in general H is NOT a normal subgroup.

Recall:

- $S^2 = SO(3)/SO(2)$
- $S^3 = SO(4)/SO(3)$
- $S^3 = SU(2)$ here $H = \{e\}$ effective transformation group

Choice of x_0:
Consider two different x_0 and $\tilde{x}_0$ and let $hx_0 = x_0$ for all $h \in H$.
Let $gx_0 = \tilde{x}_0$ then $ghg^{-1}\tilde{x}_0 = ghx_0 = gx_0 = \tilde{x}_0$.
Hence the stationary subgroup for $\tilde{x}_0$ is $\tilde{H} := gHg^{-1}$, H is conjugate to $\tilde{H}$
$\Rightarrow \mathcal{M} = G/H \simeq G/\tilde{H}$
homogenous space is uniquely defined by transitive G and one stationary subgroup H.

Invariant Measure on $\mathcal{M}$:
Let $gA := \{gx | x \in A \subset \mathcal{M}\}$ arbitrary transformation of subset A in $\mathcal{M}$
A measure μ is called G-invariant measure on $\mathcal{M}$ if for all $g \in G$ and all $A \subset \mathcal{M}$

$$\mu(A) = \mu(gA)$$

This implies for a μ-measurable function f on $\mathcal{M}$ and all $g \in G$

$$\int_{\mathcal{M}} \mathrm{d}\mu(x)\, f(x) = \int_{\mathcal{M}} \mathrm{d}\mu(x)\, f(gx)$$

Connection with invariant Haar measure

$$\int_{\mathcal{M}=G/H} \mathrm{d}\mu(x)\, f(x) = \int_G \mathrm{d}g\, f(gx_0)$$

In essence $\mathrm{d}g = \mathrm{d}\mu(x)\mathrm{d}h$.

3.4 Representations of Transformation Groups

Consider $\mathcal{H} = L^2(G/H)$ being invariant under transformation, that is,

$$\psi(x) \in \mathcal{H} \qquad \Rightarrow \qquad \psi(gx) \in \mathcal{H} \qquad \forall\, g \in G$$

Unitary Representations in $\mathcal{H}$

$$\left(D(g)\psi\right)(x) := \psi(g^{-1}x)$$

or

$$\langle x | D(g)\psi \rangle = \langle g^{-1}x | \psi \rangle = \psi(g^{-1}x)$$

Representation:

$$\left(D(g_1)D(g_2)\psi\right)(x) = \langle g_2^{-1}g_1^{-1}x|\psi\rangle = \psi((g_1g_2)^{-1}x) = (D(g_1g_2)\psi)(x)$$

Unitarity: scalar product via G-invariant measure on G/H

$$\langle\psi_1|\psi_2\rangle = \int_{G/H} \mathrm{d}\mu(x)\, \psi_1^*(x)\psi_2(x) = \int_{G/H} \mathrm{d}\mu(x)\, \psi_1^*(g^{-1}x)\psi_2(g^{-1}x) = \langle D(g)\psi_1|D(g)\psi_2\rangle$$

Comments:

- For $H = \{e\}$ $G \simeq \mathcal{M} \Rightarrow D(g)$ is (left) regular represnetation

$$\mathcal{H} = \sum_{\text{all UIR } j} d_j\, \mathcal{H}^j \qquad d_j = \dim \mathcal{H}^j$$

- General Case

$$D(g) = \sum_{\ell\in\Lambda} D^\ell(g)\,, \qquad \mathcal{H} = \sum_{\ell\in\Lambda} \mathcal{H}^\ell\,, \qquad \dim\mathcal{H}^\ell = d_\ell = \dim D^\ell$$

  Λ: Set of all *class 1 representations*, appear with multiplicity 1 in $\mathcal{H}$.

Known Example: $\mathcal{H} = L^2(S^2) = \sum_{\ell=0}^{\infty} \mathcal{H}^\ell$, $\dim\mathcal{H}^\ell = 2\ell+1$

$\mathcal{H}^\ell = \operatorname{span}\{|\ell m\rangle | m = -\ell, \ldots .\ell\}\,, \qquad L_z|\ell m\rangle = m|\ell m\rangle\,, \qquad L_z = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\varphi}$

Spherical harmonics $\langle\theta\varphi|\ell m\rangle = Y_{\ell\, m}(\theta,\varphi) = (-1)^m\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, \mathrm{e}^{\mathrm{i}m\varphi}$

Orthogonality relation: $\int_{S^2} \mathrm{d}^2\Omega Y_{\ell\, m}^*(\theta,\varphi) Y_{\ell'\, m'}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}\,, \qquad \mathrm{d}^2\Omega = \sin\theta \mathrm{d}\theta\mathrm{d}\varphi$

$\langle\theta\varphi|\ell\, 0\rangle = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$ independent of φ, rotations about z-axis

$D^\ell(h)|\ell 0\rangle = |\ell\, 0\rangle$ invariant under $h \in SO(2) \subset SO(3)$, rotations about z-axis

3.4.1 Representations of class 1

Let $\mathcal{H} = L^2(G/H)\,, \qquad \mathcal{H}^\ell \subset \mathcal{H}$, $\mathcal{H}^\ell$ irreducible invariant subspace with UIR $D^\ell(g)$

Definition: Exists an invariant vector $|\varphi_0\rangle \in \mathcal{H}^\ell$, that is, $D^\ell(h)|\varphi_0\rangle = |\varphi_0\rangle$ for all $h \in H$, then $D^\ell(g)$ is called *representation of class 1* (relative to H).

Definition: Exists for each class 1 representation *exact one* invariant vector in $\mathcal{H}^\ell$ then H is called *massiv* subgroup.

Comment: $|\varphi_0\rangle \in \mathcal{H}^\ell$ corresponds to $x_0 \in G/H$ with $hx_0 = x_0$ for all $h \in H$.

Let us choose basis in $\mathcal{H}^\ell$: $\{|\varphi_0\rangle, |\varphi_1\rangle, \ldots, |\varphi_{d_\ell-1}\rangle\}$

Representation matrices: $D^\ell_{mn}(g) := \langle\varphi_m|D^\ell(g)|\varphi_n\rangle$

In particular: For all $h, h_1, h_2 \in H$

$$D^\ell_{m0}(gh) = \langle\varphi_m|D^\ell(gh)|\varphi_0\rangle = D^\ell_{m0}(g)$$
$$D^\ell_{00}(h_1^{-1}gh_2) = \langle\varphi_0|D^\ell(h_1^{-1}gh_2)|\varphi_0\rangle = D^\ell_{00}(g)$$

Comment

- Associate spherical functions $:\Leftrightarrow f(gh) = f(g)$
- Zonal spherical functions $:\Leftrightarrow f(h_1gh_2) = f(g)$

Known Example: $G = SO(3),\ \mathcal{M} = S^2,\ \ \vec{e}(\theta,\varphi) = g\vec{e}_z$

$$D^\ell_{m0}(g) = \langle\varphi_m|D^\ell(g)|\varphi_0\rangle = \sqrt{\tfrac{4\pi}{2\ell+1}}\, Y_{\ell m}(\theta,\varphi)$$
$$D^\ell_{00}(g) = \langle\varphi_0|D^\ell(g)|\varphi_0\rangle = \sqrt{\tfrac{4\pi}{2\ell+1}}\, P_\ell(\cos\theta)$$

Orthogonality of UIR matrix elements: continuous version

$$\boxed{\int_G \mathrm{d}g\, D^\ell_{mn}(g) D^{k\,*}_{sr}(g) = \frac{1}{d_\ell}\delta_{\ell k}\delta_{ms}\delta_{nr}}$$

*** End of Lecture 3 ***