

2.3 Representations of an Abstract Group

Let V be a d -dimensional linear vector space (real or complex).

That is for $\vec{u}, \vec{v} \in V \Rightarrow \vec{u} + \vec{v} \in V$ and $\alpha\vec{v} \in V$, $\alpha \in \mathbb{R}$ or \mathbb{C} with

$$\begin{aligned} \vec{v} + \vec{u} &= \vec{u} + \vec{v} & \alpha(\beta\vec{v}) &= (\alpha\beta)\vec{v} \\ (\vec{v} + \vec{u}) + \vec{w} &= \vec{v} + (\vec{u} + \vec{w}) & 1\vec{v} &= \vec{v}, 0\vec{v} = \vec{0}, -1\vec{v} = -\vec{v} \\ \vec{v} + \vec{0} &= \vec{v} & (\alpha + \beta)\vec{v} &= \alpha\vec{v} + \beta\vec{v} \\ \vec{v} + (-\vec{v}) &= \vec{0} & \alpha(\vec{v} + \vec{u}) &= \alpha\vec{v} + \alpha\vec{u} \end{aligned}$$

Let $\mathcal{D} : V \rightarrow V$ be a linear invertible transformation (operator) acting on V

$$\mathcal{D}(\alpha\vec{u} + \beta\vec{v}) = \alpha\mathcal{D}\vec{u} + \beta\mathcal{D}\vec{v}$$

Definition: A d -dimensional linear representation of a group G is a group homomorphism

$$\mathcal{D} : \begin{array}{l} G \rightarrow GL(V) := \text{group of linear invertible transformations acting on } V \\ g \mapsto \mathcal{D}(g) \end{array}$$

with group law

$$\mathcal{D}(g_1 g_2) = \mathcal{D}(g_1) \mathcal{D}(g_2)$$

Remarks:

- Usually for finite-dimensional reps $GL(V) = GL(d, \mathbb{C})$ set of linear complex-valued $d \times d$ matrices
- $d = \infty$ is allowed, for example $V = L^2(\mathbb{R}^3)$ Hilbert space $\Rightarrow GL(V)$ is set of linear operators acting on V
- V is called *representation space*
- Let $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_d\}$ be *complete orthonormal basis* in V with scalar product

$$(\vec{e}_i, \vec{e}_j) = \delta_{ij}$$

Then

$$D_{ij}(g) := (\vec{e}_i, \mathcal{D}(g)\vec{e}_j)$$

are the matrix elements of the *matrix representation* $D(g)$.

Often no difference is made between $\mathcal{D}(g)$ and $D(g)$

- Is $D(g)$ linear operator $\forall g \in G \Leftrightarrow$ *linear representation*
Non-linear representations are also called *realisations*
- Exists a similarity transformation S such that

$$\tilde{D}(g) := S^{-1}D(g)S \quad \forall g \in G$$

is also a representation of G , then \tilde{D} and D are called *equivalent* representations (change of basis).

- Notation: $D^d(g)$ usually stands for a d -dimensional representation, $\{D_i(g)\}$ or $\{D^i(g)\}$ stands for set of reps. enumerated by an index i . Known example for rotation group is $\ell = 0, 1, 2, 3, \dots$ with dimension $d_\ell = 2\ell + 1$.

Unitary representation:

$$\begin{aligned} D(g) \text{ unitary } \forall g \in G & \quad :\Leftrightarrow \quad (D(g)\vec{u}, D(g)\vec{v}) = (\vec{u}, \vec{v}) \quad \forall g \in G \text{ and } \forall \vec{u}, \vec{v} \in V \\ & \Rightarrow D(g^{-1}) = D^\dagger(g) = D^{-1}(g) \end{aligned}$$

Faithful representation: Homomorphism is an isomorphism

$$g_1 \neq g_2 \quad \Rightarrow \quad D(g_1) \neq D(g_2)$$

Trivial representation: unitary but not faithful

$$D^{\text{trivial}}(g) := 1 \quad \forall g \in G$$

Regular representation: $G = \{g_1, g_2, \dots, g_n\}$ finite

$$gg_j =: \sum_{i=1}^n D_{ij}^{\text{reg}}(g) g_i$$

$D^{\text{reg}}(g)$ is an $n \times n$ matrix with a single 1 and rest zeros in each row and column
 n -dimensional faithful representation (group table)

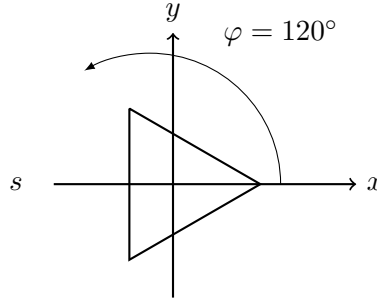
2.4 Representations of D_3

Recall: $D_3 = \{e, d, d^2, s, sd, sd^2\}$, $d^3 = e = s^2$, $sd = d^{-1}s$

- $D_s^1(g) := 1$ 1-dimensional symmetric reps. = trivial reps.
- $D_a^1(g)$ 1-dimensional anti-symmetric reps. with

$$D_a^1(g) := \begin{cases} 1 & g \in \{e, d, d^2\} = E \\ -1 & g \in \{s, sd, sd^2\} = D \end{cases}$$

- $D^2(g)$ 2-dimensional reps. explicitly constructed for generators d, s on \mathbb{R}^2



Obviously:

$$D^2(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D^2(d) = \begin{pmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$D^2(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Proof:

$$D^2(d^2) = D^2(d)D^2(d) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = [D^2(d)]^\dagger = D^2(d^{-1}) \text{ unitary}$$

$$D^2(d^3) = D^2(d)D^2(d)D^2(d) = D^2(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D^2(sd) = D^2(s)D^2(d) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = D^2(d^{-1})D^2(s) = D^2(d^{-1}s)$$

$$D^2(sd^2) = D^2(sd)D^2(d) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = D^2(d^{-1})D^2(d^{-1}s) = D^2(d^{-1})D^2(s) = D^2(d^{-2}s) \\ \Rightarrow 2\text{-dim. faithful and unitary reps.}$$

Comment: 3 conjugacy classes \Rightarrow 3 unitary irreducible reps. (see later)

2.5 Properties of Representations for Finite Groups

Maschke's Theorem:

Each representation of a finite group is equivalent to a unitary representation.

Proof: See Tutorial

Comment: Can be extended to continuous (uni-modular) groups with invariant Haar measure. In physics we usually deal with unitary irreducible representations **UIR**.

Reducible Representation:

Let $D(g)$ be a d -dimensional reps. in V , $\dim V = d$.

If there exists an invariant subspace $U \subset V$ with $\dim U < \dim V$, that is, with $\vec{u} \in U \Rightarrow D(g)\vec{u} \in U$ for all $g \in G$, then the representation is called *reducible*.

The representation matrices are of the form

$$D(g) = \left(\begin{array}{c|c} D_1(g) & R(g) \\ \hline 0 & D_2(g) \end{array} \right)$$

Irreducible Representation:

If there exists NO invariant subspace in V the representation is called *irreducible*.

Theorem:

Let $D(g)$ be unitary and reducible with invariant subspace U . Then U^\perp is also invariant subspace and $V = U \oplus U^\perp$. That is $R(g) = 0$ for unitary reducible reps.

Proof: Let $\vec{u} \in U$ and $\vec{w} \in U^\perp$ then for all $g \in G$ $D(g)\vec{u} \in U$

$$\Rightarrow 0 = (D(g)\vec{u}, \vec{w}) = (\vec{u}, D^\dagger(g)\vec{w}) = (\vec{u}, D(g^{-1})\vec{w}) \text{ for all } g \in G$$

$$\Rightarrow (\vec{u}, D(g)\vec{w}) = 0 \text{ for all } g \in G$$

Conclusion: Representation matrices of unitary reducible reps. are (in a proper basis) block-diagonal

$$D(g) = \left(\begin{array}{c|c} D_1(g) & 0 \\ \hline 0 & D_2(g) \end{array} \right) \quad \text{or} \quad D = D_1 \oplus D_2$$

Fully reducible representations:

Can the representation space of a reducible representation D be decomposed into invariant irreducible subspaces then D is called *fully reducible*

$$D = r_1 D_1 \oplus r_2 D_2 \oplus \cdots \oplus r_s D_s$$

Here $r_i \in \mathbb{N}$ denotes the multiplicity of occurrence of irr. reps. D_i in D

The representation matrices are block-diagonal

$$D(g) = \left(\begin{array}{c|c|c|c} D_1(g) & 0 & \cdots & \\ \hline 0 & D_1(g) & 0 & \ddots \\ \hline \vdots & 0 & D_2(g) & 0 \\ \hline & \vdots & 0 & \ddots \end{array} \right)$$

Comments:

Unitary reps. are either irreducible or fully reducible.

All reps. of finite groups are either irreducible or fully reducible.

Example: The *natural* reps. of $S_3 = \{e, a, b, c, d, f\}$

Recall:

$$\begin{aligned} e &:= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, & a &:= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, & b &:= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \\ c &:= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, & d &:= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, & f &:= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}. \end{aligned}$$

Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the natural basis of \mathbb{R}^3 and $P := \begin{pmatrix} 1 & 2 & 3 \\ \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$ then

$$D^{\text{nat}}(P) := \sum_{i=1}^3 \vec{e}_{\pi_i} \vec{e}_i^T \quad \text{permutation of base vectors} \quad \vec{e}_i \rightarrow \vec{e}_{\pi_i}$$

Explicit

$$\begin{aligned} D^{\text{nat}}(e) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & D^{\text{nat}}(a) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & D^{\text{nat}}(b) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\ D^{\text{nat}}(c) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & D^{\text{nat}}(d) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & D^{\text{nat}}(f) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

1-dim. invariant subspace: $\vec{v} := \frac{1}{\sqrt{3}}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3)$ obvious

2-dim. subspace orthogonal to \vec{v} :

$$\vec{u}_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{u}_2 := \vec{v} \times \vec{u}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \quad \text{obvious}$$

Change of basis:

$$S := (\vec{u}_1, \vec{u}_2, \vec{v}) = \begin{pmatrix} 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \quad \text{obviously } S^\dagger S = 1$$

Equivalent reps.:

$$\tilde{D}^{\text{nat}}(a) := S^\dagger D^{\text{nat}}(a) S = \left(\begin{array}{cc|c} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array} \right) = D^2(d) \oplus D_s^1(d)$$

$$\tilde{D}^{\text{nat}}(c) := S^\dagger D^{\text{nat}}(c) S = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right) = D^2(s) \oplus D_s^1(s)$$

Remember $S_3 \simeq D_3$ with $a \simeq d$ and $c \simeq s$

Hence $D^{\text{nat}} = D^2 \oplus D_s^1$

2.6 First and Second Schur Lemma

2.6.1 First Lemma

1. Schur Lemma:

Let D be an *irreducible* matrix representation of a group G in representation space V and M be a matrix representing an operator in V such that

$$MD(g) = D(g)M \quad \forall g \in G$$

then

$$M = \lambda \mathbf{1}, \quad \lambda = \text{const.}$$

That is, if M has G -symmetry then it is proportional to the unit matrix $\mathbf{1}$ in V .

Proof:

Let $\vec{x} \in V$ be eigenvector of M , $M\vec{x} = \lambda\vec{x}$

$\Rightarrow \vec{x}_g := D(g)\vec{x}$ is also eigenvector with same eigenvalue for all $g \in G$ as $[M, D(g)] = 0$

$\Rightarrow \exists U_\lambda \subseteq V$ such that $D(g)U_\lambda = U_\lambda$ is invariant subspace for eigenvalue λ

But D is irreducible and therefore $U_\lambda = V \Rightarrow M = \lambda\mathbf{1}$ in V

Comments:

- If the only matrix commuting with all $D(g)$ is proportional to the unit matrix then D is irreducible reps.
- All irreducible reps. of abelian groups are 1-dimensional
 D irreducible $\Rightarrow D(g_i)D(g) = D(g)D(g_i)$ for all $g, g_i \in G$. So let $M = D(g_i) \Rightarrow D(g_i) = \lambda\mathbf{1}$ and irreducible \Rightarrow 1-dimensional
- Unitary irreducible representations (UIR) of abelian groups are of the form

$$D(g) = e^{i\alpha(g)}, \quad \alpha : \begin{matrix} G \rightarrow [0, 2\pi[\\ g \mapsto \alpha(g) \end{matrix} \quad \text{with} \quad \alpha(g_1 g_2) = \alpha(g_1) + \alpha(g_2) \bmod 2\pi$$

2.6.2 Second Lemma

2. Schur Lemma:

Let D^1 and D^2 be non-equivalent UIR of dimension d_1 and d_2 . Then any rectangular $d_1 \times d_2$ matrix M which obeys

$$MD^1(g) = D^2(g)M \quad \forall g \in G$$

is the null matrix

$$M = 0$$

Proof:

Consider adjoint equation $D^{1\dagger}(g)M^\dagger = M^\dagger D^{2\dagger}(g)$ then

$D^1(g^{-1})M^\dagger = M^\dagger D^2(g^{-1}) \Rightarrow D^1(g)M^\dagger = M^\dagger D^2(g)$ for all $g \in G$

$\Rightarrow MD^1(g)M^\dagger = D^2(g)MM^\dagger = MM^\dagger D^2(g)$

1. Lemma $\Rightarrow MM^\dagger = \lambda\mathbf{1}$

Case $d_1 = d_2$: Let $\det M \neq 0$, then there exist a M^{-1} such that $D^1(g) = M^{-1}D^2(g)M$

$\Rightarrow D^1$ and D^2 are equivalent, which contradicts assumption $\Rightarrow \det M = 0$

$\Rightarrow \det MM^\dagger = |\lambda|^{d_2} = 0 \Rightarrow \lambda = 0 \Rightarrow MM^\dagger = 0 \Rightarrow M = 0$

Case $d_1 < d_2$ (without loss of generality): Complete M to $d_2 \times d_2$ matrix $\tilde{M} := (M|0)$ with additional zero columns $\Rightarrow \tilde{M}\tilde{M}^\dagger = (M|0) \begin{pmatrix} M^\dagger \\ 0 \end{pmatrix} = MM^\dagger = \lambda\mathbf{1} \Rightarrow \tilde{M} = 0 \Rightarrow M = 0$.

Both Schur lemmata in a nut shell:

For UIR

$$\boxed{\begin{array}{l} MD^i(g) = D^j(g)M \quad \Rightarrow \quad M = \lambda \delta_{ij} \\ \text{with} \quad \delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \text{ inequivalent reps.} \\ 1 & \text{for } i = j \text{ equivalent reps.} \end{cases} \end{array}}$$

2.6.3 Application to Eigenvalue Problems

Let H be linear operator in V . For example: $H = \vec{P}^2/2m + V(|\vec{Q}|)$, $V = L^2(\mathbb{R}^3)$.

Let D be unitary reducible reps of G in V . For example $G = SO(3)$, $D(g(\vec{\omega})) = \exp\{-i\vec{\omega} \cdot \vec{L}\}$.

We say H has G -symmetry if

$$[D(g), H] = 0 \quad \forall g \in G \quad \Leftrightarrow \quad D(g)H = HD(g)$$

Problem: Find eigenvalues and eigenvectors of H in V , $|H - \lambda \mathbf{1}| = 0$.

Note: D is completely reducible

$$D(g) = c_1 D^1(g) + c_2 D^2(g) + \dots + c_n D^n(g), \quad n \leq \infty$$

$D^i(g)$: UIR of dimension d_i

c_i : multiplicity of D^i in D

Reduction of Problem:

With suitable basis in V the reps. matrix for D is block diagonal

Example: $D(g) = 2 D^1(g) + D^2(g)$

$$D(g) = \left(\begin{array}{cc|c} D^1(g) & 0 & 0 \\ 0 & D^1(g) & 0 \\ 0 & 0 & D^2(g) \end{array} \right) \begin{matrix} \Sigma^1 \\ \Sigma^2 \end{matrix}$$

Conglomerate: Σ^i inv. subspace of V containing all the UIR D^i .

Write H in that basis

$$H = \left(\begin{array}{cc|c} H_{11}^{(1)} & H_{12}^{(1)} & H_{11}^{(12)} \\ H_{21}^{(1)} & H_{22}^{(1)} & H_{21}^{(12)} \\ H_{11}^{(21)} & H_{22}^{(21)} & H_{11}^{(2)} \end{array} \right) \begin{matrix} \Sigma^1 \\ \Sigma^2 \end{matrix}$$

In general:

$H_{lm}^{(i)} = d_i \times d_i$ matrix, lm element of a submatrix of H in Σ^i

$l = m$ submatrix of H in subspace belonging to a fixed D^i

$l \neq m$ overlap of m -th and l -th UIR D^i , l and $m \in \{1, 2, \dots, c_i\}$

$H_{lm}^{(ij)} = d_i \times d_j$ matrix, overlap of l -th UIR D^i with m -th UIR $D^j \neq D^i$

Symmetry of H : $D(g)H = HD(g)$ for all $g \in G$

$$\Rightarrow \begin{cases} D^i(g)H_{lm}^{(i)} = H_{lm}^{(i)}D^i(g) & \xrightarrow{1.\text{SL}} H_{lm}^{(i)} = h_{lm}^{(i)}\mathbf{1}_{d_i} \\ D^i(g)H_{lm}^{(ij)} = H_{lm}^{(ij)}D^j(g) & \xrightarrow{2.\text{SL}} H_{lm}^{(ij)} = 0 \end{cases}$$

In our example:

$$H = \left(\begin{array}{cc|c} h_{11}^{(1)}\mathbf{1}_{d_1} & h_{12}^{(1)}\mathbf{1}_{d_1} & 0 \\ h_{21}^{(1)}\mathbf{1}_{d_1} & h_{22}^{(1)}\mathbf{1}_{d_1} & 0 \\ 0 & 0 & h_{11}^{(2)}\mathbf{1}_{d_2} \end{array} \right)$$

In subspace Σ^i operator H consists of $c_i \times c_i$ blocks of dimension d_i being diagonal matrices.
In general with definition $\left(\tilde{H}^{(i)}\right)_{lm} := h_{lm}^{(i)} \Rightarrow \tilde{H}^{(i)}$ is $c_i \times c_i$ matrix

$$\Rightarrow H = \sum_{i=1}^n \left(\tilde{H}^{(i)} \times \mathbf{1}_{d_i} \right) \quad V = \bigoplus_{i=1}^n V_i, \quad \dim V_i = c_i d_i$$

In our example:

$$H = \left(\begin{array}{c|c} \tilde{H}^{(1)} \times \mathbf{1}_{d_1} & 0 \\ \hline 0 & \tilde{H}^{(2)} \times \mathbf{1}_{d_2} \end{array} \right)$$

with

$$\tilde{H}^{(1)} = \begin{pmatrix} h_{11}^{(1)} & h_{12}^{(1)} \\ h_{21}^{(1)} & h_{22}^{(1)} \end{pmatrix}, \quad c_1 = 2 \quad \text{and} \quad \tilde{H}^{(2)} = h_{11}^{(2)}, \quad c_2 = 1$$

Conclusion:

With a suitable basis (also within the subspaces Σ^i) the eigenvalue problem for H can be reduced to n eigenvalue problems of the form

$$\left| \tilde{H}^{(i)} - \lambda^{(i)} \mathbf{1}_{c_i} \right| = 0$$

n = number of different UIR of symmetry of H occurring in V .

Comments:

- $D(g) = c D^{\text{trivial}}(g)$, $c = \dim V$ only trivial representation
 \Rightarrow no symmetry \Rightarrow no simplification
- $D(g) = D^i(g)$ is already UIR $\Rightarrow H = \lambda \mathbf{1}$, only one eigenvalue, problem solved
- D^i appears only once in decomposition, that is, $c_i = 1$, then invariant subspace Σ^i is also eigenspace of $H \Rightarrow$ degeneracy due to symmetry
Same eigenvalue may accidentally occur also in other subspaces
 \Rightarrow accidental degeneracy (usually a sign of an additional hidden symmetry)

Summary:

1. Choose suitable symmetry group and its representation in V
Aim is to have c_i 's as small as possible $\Rightarrow d_i$ as large as possible as $\dim V = \sum_{i=1}^n c_i d_i$
higher symmetry groups have higher-dim. UIR ($C_n \subset D_n \subset S_n$)
 \Rightarrow "higher" simplification
2. Decompose representation into UIRs
3. Choose symmetry adopted basis in subspaces Σ

Know example from QM

$$H = \frac{\vec{P}^2}{2m} + V(|\vec{Q}|), \quad V = L^2(\mathbb{R}^3) = L^2(\mathbb{R}^+) \otimes L^2(S^2)$$

$$L^2(S^2) = \bigoplus_{l=0}^{\infty} D^l, \quad d_l = 2l + 1, \quad c_l = 1$$

$$D(g) = \exp\{-i\vec{\omega} \cdot \vec{L}\} \text{ with } [H, D(g)] = 0 \text{ for all } g \in SO(3) \text{ as } [H, \vec{L}] = \vec{0}.$$

$$\Rightarrow H = \sum_{l=0}^{\infty} H_r^l \otimes \mathbf{1}_{2l+1} \text{ with}$$

$$H_r^l = -\frac{\hbar^2}{2m} \partial_r^2 + \frac{\hbar^2 l(l+1)}{2m} + V(r)$$

in suitable basis

$$\Psi_l(\vec{r}) = \sum_{l=0}^{\infty} r R_l(r) \sum_{l=-m}^m Y_{lm}(\theta, \varphi)$$

Note $\langle \theta \varphi | lm \rangle = Y_{lm}(\theta, \varphi)$ and $D^l = \sum_{m=-l}^m |lm\rangle \langle lm|$

An explicit example will be worked out in the Tutorial in Exercise 4.

2.7 Orthogonality of Representations and Characters

2.7.1 Orthogonality of UIR

Theorem: Let $D^i(g)$ and $D^j(g)$ be matrices of two UIR of dimension d_i and d_j for a group G , $g \in G$. Then the following orthogonality relation of the matrix elements holds:

$$\boxed{\frac{1}{n} \sum_{g \in G} D_{\mu\nu}^i(g) D_{\rho\sigma}^{j*}(g) = \frac{1}{d_i} \delta_{ij} \delta_{\mu\rho} \delta_{\nu\sigma}},$$

where $n = \text{ord } G$ and

$$\delta_{ij} = \begin{cases} 0 & i \neq j \text{ inequivalent reps.} \\ 1 & i = j \text{ equivalent reps.} \end{cases}$$

Proof: Consider

$$M := \sum_{g \in G} D^i(g) X D^j(g^{-1})$$

with X being an arbitrary $d_i \times d_j$ matrix. Then for all $g_0 \in G$

$$\begin{aligned} D^i(g_0)M &= \sum_{g \in G} D^i(g_0) D^i(g) X D^j(g^{-1}) D^j(g_0^{-1}) D^j(g_0) \\ &= \sum_{g \in G} D^i(g_0 g) X D^j((g g_0)^{-1}) D^j(g_0) \\ &= M D^j(g_0) \end{aligned}$$

and therefore $M = \lambda \mathbf{1} \delta_{ij}$, see 1. and 2. Schur lemma. On the other hand we have

$$M_{\mu\rho} = \sum_{g \in G} \sum_{r,s} D_{\mu r}^i(g) X_{rs} D_{s\rho}^j(g^{-1})$$

Let us choose $X_{rs} = \delta_{r\nu} \delta_{s\sigma}$ then

$$M_{\mu\rho} = \sum_{g \in G} D_{\mu\nu}^i(g) D_{\sigma\rho}^j(g^{-1}) = \lambda \delta_{\mu\rho} \delta_{ij}$$

Now we calculated λ for $i = j$ by setting $\mu = \rho$ and sum over μ .

$$\lambda d_i = \sum_g \sum_{\mu=1}^{d_i} D_{\mu\nu}^i(g) D_{\sigma\mu}^i(g^{-1}) = \sum_g \underbrace{D_{\sigma\nu}^i(g^{-1}g)}_{\delta_{\sigma\nu}} \stackrel{=e}{=} n \delta_{\sigma\nu}$$

Hence, $\lambda = \frac{n}{d_i} \delta_{\sigma\nu}$ and we conclude

$$\boxed{\frac{1}{n} \sum_{g \in G} D_{\mu\nu}^i(g) D_{\sigma\rho}^j(g^{-1}) = \frac{1}{d_i} \delta_{ij} \delta_{\mu\rho} \delta_{\nu\sigma}} \quad (\text{I})$$

Comments:

- (I) is valid for irreducible reps not necessarily unitary
- For unitary reps follows $D_{\sigma\rho}^j(g^{-1}) = D_{\rho\sigma}^{j*}(g)$ and proof is completed
- Extension to compact groups obvious

$$\frac{1}{n} \sum_g (\cdot) \Rightarrow \int_G dg (\cdot)$$

- With proper interpretation also to non-compact uni-modular groups having $d_i = \infty$ for UIR.

Examples

- $G = U(1)$ then $D^m(g) = e^{im\varphi}$, $d_m = 1$, $m \in \mathbb{Z}$, $g = g(\varphi)$
 $\int_{U(1)} dg D^m(g) D^{n*}(g) = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{im\varphi} e^{-in\varphi} = \delta_{mn}$
- $G = T^1$ then $D^k(g) = e^{-ikx}$, $k \in \mathbb{R}$, $g = g(x)$, $x \in \mathbb{R}$ (see homework)
 $\int_{T^1} dg D^k(g) D^{k'*}(g) = \int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{-ikx} e^{-k'x} = \delta(k - k')$

2.7.2 Abstract harmonic analysis

Consider finite group $G = \{g_1, g_2, \dots, g_n\}$, $n = \text{ord } G$ and a well-defined function

$$f : \begin{array}{l} G \rightarrow \mathbb{C} \\ g \mapsto f(g) \end{array}$$

Let

$$\vec{f} := (f(g_1), f(g_2), \dots, f(g_n))$$

be element of vector space $V \simeq \mathbb{C}^n$ with scalar product

$$\langle \vec{h}, \vec{f} \rangle := \frac{1}{n} \sum_{g \in G} h^*(g) f(g),$$

then for an arbitrary UIR

$$\vec{e}_{\mu\nu}^i := \sqrt{d_i} (D_{\mu\nu}^i(g_1), D_{\mu\nu}^i(g_2), \dots, D_{\mu\nu}^i(g_n))^T$$

obeys

$$\langle \vec{e}_{\rho\sigma}^j, \vec{e}_{\mu\nu}^i \rangle = \frac{\sqrt{d_i d_j}}{n} \sum_{g \in G} D_{\rho\sigma}^{j*}(g) D_{\mu\nu}^i(g) = \delta_{ij} \delta_{\mu\rho} \delta_{\nu\sigma}.$$

That is, it forms a (complete) orthonormal set in V .

Comment: For a fixed i there exist d_i^2 linearly independent unit vectors $\Rightarrow \sum_i d_i^2 \leq n$. In other words, for finite groups there exist only a finite number of UIR.

Theorem of Burnside:

$$\sum_{\text{all UIR}} d_i^2 = n$$

Proof: Later

Conclusion: $\{\vec{e}_{\mu\nu}^i\}$ forms a complete set in $V \simeq \mathbb{C}^n$

$$\vec{f} = \sum_{i, \mu, \nu} d_i \left\langle \frac{1}{\sqrt{d_i}} \vec{e}_{\mu\nu}^i, \vec{f} \right\rangle \frac{1}{\sqrt{d_i}} \vec{e}_{\mu\nu}^i$$

with

$$\tilde{f}_{\nu\mu}^i := \left\langle \frac{1}{\sqrt{d_i}} \tilde{e}_{\mu\nu}^i, \vec{f} \right\rangle = \frac{1}{n} \sum_g D_{\mu\nu}^{i*}(g) f(g) = \frac{1}{n} \sum_g D_{\nu\mu}^i(g^{-1}) f(g).$$

Or for a fixed component $f(g)$ of \vec{f}

$$\begin{aligned} f(g) &= \sum_{\text{all UIR } i} d_i \sum_{\mu, \nu=1}^{d_i} \tilde{f}_{\nu\mu}^i D_{\mu\nu}^i(g) \\ \tilde{f}_{\nu\mu}^i &= \frac{1}{n} \sum_{g \in G} f(g) D_{\nu\mu}^i(g^{-1}) \end{aligned}$$

Above decomposition is called abstract Fourier or harmonic analysis.

Peter-Weyl-Theorem:

$$\begin{aligned} f(g) &= \sum_{\text{all UIR } i} d_i \text{Tr} \left(\tilde{f}^i D^i(g) \right) \\ \tilde{f}^i &= \frac{1}{n} \sum_{g \in G} f(g) D^i(g^{-1}) \end{aligned}$$

Comments:

- Parseval equation (without proof)

$$\frac{1}{n} \sum_{g \in G} |f(g)|^2 = \sum_{\text{all UIR } i} \sum_{\mu, \nu=1}^{d_i} d_i |\tilde{f}_{\nu\mu}^i|^2$$

- Extension to compact groups and with proper interpretation even to uni-modular groups possible

Examples:

- $G = Z_2$: See Homework Problem 1
- $G = U(1)$: Fourier series

$$f(\varphi) = \sum_{m \in \mathbb{Z}} \tilde{f}^m e^{-im\varphi}, \quad \tilde{f}^m = \frac{1}{2\pi} \int_0^{2\pi} d\varphi f(\varphi) e^{im\varphi}$$

- $G = T^1$: Fourier analysis

$$f(x) = \int_{\mathbb{R}} dk \tilde{f}(k) e^{-ikx}, \quad \tilde{f}(k) = \frac{1}{2\pi} \int_{\mathbb{R}} dx f(x) e^{ikx}$$

*** End of Lecture 2 ***