Group Theory for Physicists

Lecture Notes

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Preliminaries

Dates:
Five Mondays 28.04.25, 05.05.25, 12.05.25, 19.05.25, 26.05.22, 02.06.25
Lecture 9 - 12, Tutorial 14 - 16, Homework Problems
Test TBD
Script and other details are available on StudOn and at
`https://www.eso.org/~gjunker/VorlesungSS2025.html`

Literature:
Any group theory textbook will cover most of the topics. Some elementary ones are

- W. Lucha and F.F. Schöberl, *Gruppentheorie* (BI, 1993)
- H.F. Jones, *Groups, Representations and Physics* 2nd Ed. (Taylor & Francis,1998)
- E. Stiefel and A. Fässler, *Gruppentheoretische Methoden und ihre Anwendung* (Teuber, 1979)
- ...

Group theory:
Is the mathematical tool to describe symmetries, for example, in physical systems. Thus group theory and the closely related representation theory have many important applications in physics, chemistry, and materials science.

Aim of lecture:
Present the basic concepts of group theory enabling us to utilise symmetries of physical systems to analyse their properties.
Here focus on quantum mechanics and statistical physics.

1 Basic Terms and Definitions

1.1 Definition of an Abstract Group

Definition: A *group* G, or better $(G, \circ)$, is a set of elements (finite or infinite in number),

$$G = \{g_1, g_2, \ldots\} \qquad \text{or} \qquad G = \{g(\alpha) | \alpha \in I\}, \quad I = \text{ index set}$$

with a *composition law* (group multiplication)

$$\circ : \begin{array}{l} G \times G \to G \\ (g_1, g_2) \mapsto g_1 \circ g_2 \end{array}$$

satisfying below conditions

1. *Associative Law:*
$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 = g_1 \circ g_2 \circ g_3$$
2. *Unit Element:* $\exists\, e \in G$ such that
$$e \circ g = g \circ e = g \qquad \forall\, g \in G$$
3. *Inverse Element:* $\forall\, g \in G\ \exists\, g^{-1} \in G$ such that
$$g^{-1} \circ g = e = g \circ g^{-1}$$

Remarks:

- In general $g_1 \circ g_2 \neq g_2 \circ g_1$, that is, the group multiplication is *not commutative* $\Leftrightarrow$: *non-abelian group*
- *Abelian group* $:\Leftrightarrow g_1 \circ g_2 = g_2 \circ g_1\ \forall\, g_1, g_2 \in G$
- *Order of a group:* Number of (inequivalent) elements
$$g = \{g_1, g_2, \ldots, g_n\} \qquad \Rightarrow \qquad \operatorname{ord} G = n$$
- *Finite group* $:\Leftrightarrow \qquad \operatorname{ord} G < \infty$
- *Discrete group:* Countable infinite number of elements
- *Continuous group* uncountable number of elements
$$g = g(\alpha)\,, \qquad \alpha \in I \qquad \text{index set}$$

Conclusions from definition:

- $g_1 \circ g = g_2 \circ g \qquad \Rightarrow \qquad g_1 = g_2$
- $g \circ g_1 = g \circ g_2 \qquad \Rightarrow \qquad g_1 = g_2$
- e and g^{-1} are unique
- $(g^{-1})^{-1} = g \qquad (g_1 \circ g_2)^{-1} = g_2^{-1} \circ g_1^{-1}$
- $g_1 \circ g = g_2$ and $g \circ g_1 = g_2$ have solutions $g = g_1^{-1} \circ g_2$ and $g = g_2 \circ g_1^{-1}$, respectively.

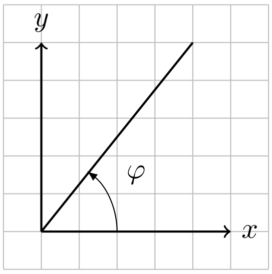
Notation: From now on

$$g_1 \circ g_2 = g_1 g_2 \qquad \underbrace{g \circ g \circ \ldots \circ g}_{\text{n}-\text{times}} =: g^n$$

Examples:

Abelian Groups

- *Trivial group:* $E = \{e\}$, $\operatorname{ord} E = 1$
- *Reflexion group:* $C_2 = \{e, \sigma\}$ with $\sigma^2 = e$, $\operatorname{ord} C_2 = 2$
- $Z_n = \{0, 1, 2, \dots, n-1\}$, $\circ =$ addition mod n, $\operatorname{ord} Z_n = n$
- $(\mathbb{Z}, +)$ and $(\mathbb{R}^+, \cdot)$, both are of infinite order
- Rotation in plane by angle $\varphi \in I = [0, 2\pi[$



Non-abelian Groups

- $GL(n, K)$: Set of invertible $n \times n$ matrices over field K, $\circ =$ matrix multiplication
- S_n : Group of permutations of n objects, $\operatorname{ord} S_n = n!$

1.2 Group Structures

1.2.1 Subgroups

Definition: A subset $H \subset G$ is called a *subgroup* of G if the group multiplication of G restricted on the subset H is closed, i.e. $\circ : H \times H \to H$

- $(\{e\}, \circ)$ and $(G, \circ)$ are *trivial* subgroups
- Non-trivial subgroups $\Leftrightarrow$: *proper* subgroups

1.2.2 Conjugation and Conjugacy Classes

Definition: Two elements $g_1, g_2 \in G$ are *conjugate* to each other if

$$\exists\, g \in G \qquad \text{such that} \qquad g_1 = g\, g_2\, g^{-1}$$

Conjugation is *transitive*: $\begin{array}{l} g_1 = g\, g_2\, g^{-1} \\ g_2 = h\, g_3\, h^{-1} \end{array} \Rightarrow g_3 = k\, g_1\, k^{-1}$

Proof: $g_3 = h^{-1}\, g_2\, h = h^{-1} g^{-1}\, g_1\, gh = k\, g_1\, k^{-1}$ with $k = (gh)^{-1}$

Definition: The set of all conjugate elements is called *conjugacy class* or simply class

Remarks:

- A class is uniquely defined by one of its elements say a

$$\{g_1 a g_1^{-1}, g_2 a g_2^{-1}, \dots, g_n a g_n^{-1}\} \qquad \text{for} \qquad \operatorname{ord} G = n$$

- $\{e\}$ is a class by itself
- Each element $g \in G$ belongs exactly to one class $\Rightarrow$ disjunct partition of G
- For abelian groups each element forms its own class. Why?

Definition: The *order of a group element* is the smallest integer $m \in \mathbb{N}$ such that $g^m = e$.

Remarks:

- All elements within one class have the same order

$$g^m = e \qquad \Rightarrow \qquad (g_i g g_i^{-1})^m = g_i g \cdots g g_i^{-1} = g_i g^m g_i^{-1} = e$$

- Let G be a matrix group $\Rightarrow \mathrm{Tr}\,(g_i g g_i^{-1}) = \mathrm{Tr}\, g \Rightarrow$ the trace is constant on a class
- Functions that are constant for members of the same conjugacy class are called class functions.

1.2.3 Normal Subgroups (Invariant Subgroups, Self-conjugate Subgroups)

Definition: A subgroup $N \subset G$ is called *normal subgroup* (or invariant subgroup or self-conjugate subgroup) if

$$\forall\, n \in N \qquad \text{and} \qquad \forall\, g \in G \qquad \Rightarrow \qquad g n g^{-1} \in N$$

In short

$$g N g^{-1} = N \qquad \forall\, g \in G$$

Remarks:

- Normal subgroups consist of classes
- *Simple group:*All its normal subgroups are trivial
- *Semi-Simple group:* All its normal subgroups are abelian

1.2.4 Cosets

Let $H \subset G$ be a subgroup of G and $g \in G$ a fixed group element

Left coset: $gH := \{gh | h \in H\}$, mainly used with terminology coset

Right coset: $Hg := \{hg | h \in H\}$

Left and Right cosets are disjoint partitions of G

In general $Hg \neq gH$

Index of H: Number of left cosets $= k =$ Number of right cosets

Lagrange's Theorem:

$$\mathrm{ord}\, H = \frac{1}{k} \mathrm{ord}\, G\,, \qquad \text{where } k \in \mathbb{N} \text{ is the index of } H$$

Proof: Disjoint partition $G = \{H, g_1 H, \ldots, g_{k-1} H\}$ and $\mathrm{ord}\,(g_i H) = \mathrm{ord}\, H$

Remark: $Hg = gH \qquad \forall\, g \in G \qquad \Leftrightarrow \qquad H$ is normal subgroup

1.2.5 Quotient Group

Let $N \subset G$ be normal subgroup of G with index k.

$$F := \{N, g_1 N, g_2 N, \ldots, g_{k-1} N\}$$

is disjunct partition of G into cosets with respect to N.
Notation: $F = G/N$ is *quotient group* with $\mathrm{ord}\, F = k$

Proof of group properties:

- Group multiplication:

$$(g_1N)(g_2N) = g_1Ng_2N = g_1g_2g_2^{-1}Ng_2N = g_1g_2NN = g_3N \in F$$

- Neutral element:

$$gN \circ N = gN\,, \qquad N \circ gN = NgN = gg^{-1}NgN = gNN = gN$$

- Inverse element:

$$g^{-1}N \circ gN = NN = N\,, \qquad gN \circ g^{-1}N = gNg^{-1}N = N$$

1.3 Group Morphisms

Group homomorphism: Let $(G, \circ)$ and $(G', \star)$ be two groups. Then the mapping

$$\Phi : \begin{array}{l} G \to G' \\ g \mapsto \Phi(g) \end{array} \qquad \text{with} \qquad \Phi(g_1) \star \Phi(g_2) = \Phi(g_1 \circ g_2)$$

is a group homomorphism. In general the mapping is not reversible.

Group isomorphism: A homomorphism with bijective mapping Φ, that is,

$$g_1 \neq g_2 \qquad \Rightarrow \qquad \Phi(g_1) \neq \Phi(g_2)\,, \text{reversible}\,,$$

$$\exists\, \Phi^{-1} : \begin{array}{l} G' \to G \\ \Phi(g) \mapsto g \end{array}$$

Isomorphic groups:

$$G_1 \simeq G_2 \qquad :\Leftrightarrow \qquad \exists \quad \text{Isomorphism} \quad \Phi : G_1 \to G_2$$

Isomorphic groups are in essence identical.
Example: $SO(2) \simeq U(1)$, rotation in plane $\simeq$ multiplication of complex number by $\mathrm{e}^{\mathrm{i}\phi}$

Automorphism: Isomorphism $G \to G$

Inner Automorphism:

$$\Phi_h : \begin{array}{l} G \to G \\ g \mapsto \Phi_h(g) := hgh^{-1} \end{array} \qquad h \in G \quad \text{fixed, conjugation}$$

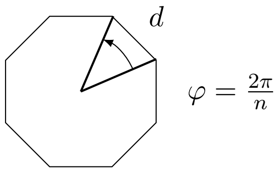
Outer Automorphism: all automorphism being not an inner automorphism

2 Finite Groups and Representations

2.1 Examples of Finite Groups and Properties

2.1.1 The cylclic group C_n

Symmetry group of rotations of a regular polygon with n directed sides



$$C_n := \{e, d, d^2, \ldots, d^{n-1}\} \qquad \text{with} \qquad d^n = e$$

Generator: $d :=$ rotation by angle $\varphi = \frac{2\pi}{n}$

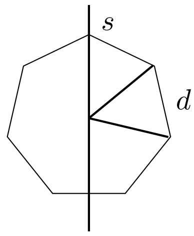
Generating set of a group: Set of group elements (generators) which allows to generate all group elements via products and inverses. In general this set is not unique.

C_n is abelian and isomorphic to Z_n (under addition of integers mod n):

$$C_n \simeq Z_n \qquad \text{as} \qquad d^r d^s = d^{r+s} \qquad \text{with} \qquad r + s = (r+s) \bmod n\,, \qquad \operatorname{ord} C_n = n$$

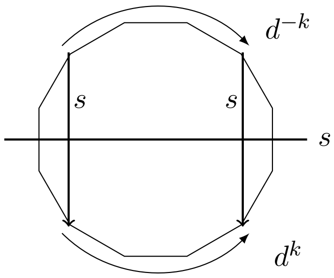
2.1.2 The dihedral group D_n (Diedergruppe)

Group of n rotations d and reflection s keeping a regular n-polygon invariant



Example D_8 : rotations (1st line) and reflections (2nd line)

Generating set: $\{d, s\}$ with $d =$ rotation by $\varphi = \frac{2\pi}{n}$ and $s =$ reflection on fixed axis
$D_n := \{e, d, d^2, \ldots, d^{n-1}, s, sd, \ldots, sd^{n-1}\}$ with $d^n = e = s^2\,,\quad d^{-k}s = sd^k\,,\quad d^k s = sd^{-k}$



$$\operatorname{ord} D_n = 2n$$

Subgroup $C_n \subset D_n$

D_n is NOT abelian for $n > 2$ as $d^{-1}s = sd \neq sd^{-1}$

2.1.3 The permutation group S_n

Group of permuations of n objects
$\text{ord}\, S_n = n!$
General element: Object $j \to \pi_j$, where $\pi_j \in \{1, 2, \ldots, n\}$ and $\pi_j \neq \pi_k$ for $j \neq k$

$$P = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$$

Neutral element:

$$e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

Inverse element:

$$P^{-1} = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

Group multiplication: successive permutation

$$\text{Example:} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\text{Proof:} \quad \begin{array}{c} 1 \to 3 \to 2 \\ 2 \to 1 \to 1 \\ 3 \to 2 \to 3 \end{array}$$

$$\text{But:} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\Rightarrow \qquad S_n \text{ is NOT abelian for } n \geq 3$$

Exercise: Show that $S_2 \simeq C_2$

More on permutation group in Tutorial Exercise 1

2.2 Cayley or Group Tables

2.2.1 Definition

G	g_1	g_2	$\cdots$	g_i	$\cdots$	g_n
g_1	g_1^2	$g_1 g_2$	$\cdots$	$g_1 g_i$	$\cdots$	$g_1 g_n$
g_2	$g_2 g_1$	g_2^2	$\cdots$	$g_2 g_i$	$\cdots$	$g_2 g_n$
$\vdots$			$\ddots$			
g_i	$g_i g_1$	$g_i g_2$	$\cdots$	g_i^2	$\cdots$	$g_i g_n$
$\vdots$					$\ddots$	
g_n	$g_n g_1$	$g_n g_2$		$\cdots$		g_n^2

Remarks:

- Useful only for finite groups of low order n
- G abelian $\Leftrightarrow$ group table is symmetric as $g_i g_j = g_j g_i$
- Isomorphic groups have identical tables

Examples:

C_2	e	d
e	e	d
d	d	e

recall $d^2 = e$

C_3	e	d	d^2
e	e	d	d^2
d	d	d^2	e
d^2	d^2	e	d

recall $d^3 = e$

2.2.2 Cayley's theorem

Theorem: Every finite group G of order n is a subgroup of S_n

Proof: Obvious as each row in group table corresponds to a rearrangement of group elements.

$$\{gg_1, gg_2, \dots, gg_n\} =: \{g_{\pi_1}, g_{\pi_2}, \dots, g_{\pi_n}\}$$

$$\Rightarrow \qquad g \to P(g) = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$$

Corollary: Each row in the group table contains each element of G exactly once.

Also known as rearrangement theorem.

Remarks:

- Number of different (non-isomorphic) groups of order n is finite
- There exists only ONE group of order 2, the reflection group $S_2 \simeq C_2 \simeq Z_2$
- $S_3 \simeq D_3$ (obvious as $\operatorname{ord} S_3 = 6 = \operatorname{ord} D_3$) has only ONE subgroup of order 3 isomorphic to $C_3 \Rightarrow$ There exists only ONE group of order 3.
- S_4 has two subgroups of order 4: $\{C_4, D_2\}$
- Group summation:
$$\sum_{g\in G} f(g) = \sum_{g\in G} f(gh) = \sum_{g\in G} f(hg) = \sum_{g\in G} f(g^{-1})$$
Is valid for all finite groups.
Extension to continuous (unimodular) groups via invariant Haar measure possible.
- Group average: As $\sum_{g\in G} 1 = \operatorname{ord} G$
$$\langle\cdot\rangle_G := \frac{1}{\operatorname{ord} G} \sum_{g\in G} (\cdot)$$

2.2.3 Klein's four group (Kleinsche Vierergruppe $V = D_2$)

Recall: $D_2 = \{e, d, s, ds\}$ with $d^2 = e = s^2$, $d = d^{-1}$, $s = s^{-1}$, $ds = sd$ abelian

D_2	e	d	s	sd
e	e	d	s	sd
d	d	e	sd	s
s	s	sd	e	d
sd	sd	s	d	e

Remarks:

- $E = \{e\}$ is trivial subgroup
- e on diagonal $\Leftrightarrow$ each element is its inverse
- To each e on diagonal exists a subgroup of order 2.
$\{e, d\}$, $\{e, s\}$ and $\{e, sd\}$ are normal subgroups isomorphic to C_2
- Factor group $D_2/C_2 = C_2$ or $D_2 = C_2 \otimes C_2$ (direct product of groups in Tutorial)
- Other representation: $\{1, 3, 5, 7\}$ with group law being multiplication modulo 8

2.2.4 The D_3 group

Recall: $D_2 = \{e, d, d^2, s, ds, sd^2\}$ with $d^3 = e = s^2$, $ds = sd^{-1} = sd^2$, $d^2 s = sd^{-2} = sd$

D_3	e	d	d^2	s	sd	sd^2
e	e	d	d^2	sd	sd	sd^2
d	d	d^2	e	sd^2	s	sd
d^2	d^2	e	d	sd	sd^2	s
s	s	sd	sd^2	e	d	d^2
sd	sd	sd^2	s	d^2	e	d
sd^2	sd^2	s	sd	d	d^2	e

Remarks:

- Subgroups: $C_3 \subset D_3$, $H_1 := \{e, s\}$, $H_2 := \{e, sd\}$, $H_3 := \{e, sd^2\}$ with $H_i \simeq C_2$
- Cosets: $\text{ord}\, D_3 = 6$, $\text{ord}\, C_3 = 3$, $\text{ord}\, C_2 = 2$
  Lagrange: Index $C_2 = 6/2 = 3$ $\rightarrow$ 3 cosets for $C_2 \simeq H_1$

  3 right cosets of H_1: $H_1 = \{e, s\}$, $H_1 d = \{d, sd\}$, $H_1 d^2 = \{d^2, sd^2\}$
  3 left cosets of H_1: $H_1 = \{e, s\}$, $dH_1 = \{d, ds\}$, $d^2 H_1 = \{d^2, d^2 s\}$
  Note: $dH_1 \neq H_1 d$, it is NOT a normal subgroup

  2 right cosets of C_3: $C_3 = \{e, d, d^2\}$, $C_3 s = \{s, ds, d^2 s\} = \{s, sd^2, sd\}$
  2 left cosets of C_3: $C_3 = \{e, d, d^2\}$, $sC_3 = \{s, sd, sd^2\}$
  Note: $sC_3 = C_3 s$, C_3 is normal subgroup, $D_3/C_3 \simeq C_2$
- Quotient group: $C_2 := \{E, D\}$, where $E := \{e, d, d^2\}$, $D := \{s, sd, sd^2\}$
  $\rightarrow ED = DE$, $D^2 = E$
- Conjugacy classes: $\{e\}$, $\{d, d^2\}$, $\{s, sd, sd^2\}$ (see Tutorial)

*** End of Lecture 1 ***