Problem 11: Generators of SO(3) on $L^2(\mathbb{R}^3)$

a) From Problem 10 we know

$$\exp\{-i\delta\alpha^a L_a\} = L_a^2\left(\cos(\delta\alpha^a)-1\right) + 1 - i L_a \sin\delta\alpha^a$$
$$\approx 1 - i\,\delta\alpha^a L_a + O((\delta\alpha^a)^2)$$

$$\rightsquigarrow\; g(\delta\vec\alpha) \approx 1 - i\sum_{a=1}^{3}\delta\alpha^a L_a + O(\delta\vec\alpha^2)$$
$$= 1 + \begin{pmatrix} 0 & -\delta\alpha^3 & \delta\alpha^2 \\ \delta\alpha^3 & 0 & -\delta\alpha^1 \\ -\delta\alpha^2 & \delta\alpha^1 & 0 \end{pmatrix} + O(\delta\vec\alpha^2)$$

b)

$$\delta\vec x = \vec x' - \vec x = \begin{pmatrix} 0 & -\delta\alpha^3 & \delta\alpha^2 \\ \delta\alpha^3 & 0 & -\delta\alpha^1 \\ -\delta\alpha^2 & \delta\alpha^1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} -x_2\,\delta\alpha^3 + x_3\,\delta\alpha^2 \\ x_1\,\delta\alpha^3 - x_3\,\delta\alpha^1 \\ -x_1\,\delta\alpha^2 + x_2\,\delta\alpha^1 \end{pmatrix} =: \begin{pmatrix} \delta x^1 \\ \delta x^2 \\ \delta x^3 \end{pmatrix}$$

$$\rightsquigarrow\; X_1 = -\sum_{k=1}^{3}\frac{\delta x_k}{\delta\alpha^1}\partial_k = x_3\partial_2 - x_2\partial_3$$

$$X_2 = -\sum_{k=1}^{3}\frac{\delta x_k}{\delta\alpha^2}\partial_k = x_1\partial_3 - x_3\partial_1$$

$$X_3 = -\sum\frac{\delta x_k}{\delta\alpha^3}\partial_k = x_2\partial_1 - x_1\partial_2$$

$$\rightsquigarrow\; \vec X = -\vec x\times\vec\partial\,,\qquad \vec p = \frac{\hbar}{i}\vec\partial$$

$$\underline{\underline{\vec X = \frac{1}{i\hbar}\vec L}}\qquad \text{with } \vec L = \vec x\times\vec p$$

c) $X_i = -\varepsilon_{ijk} x_j \partial_k$

$\rightsquigarrow$

$$\underline{[X_i, X_j]} = \varepsilon_{ilm} [x_l \partial_m, x_k \partial_n] \varepsilon_{jkn}$$

$$[x_l \partial_m, x_k \partial_n] = x_l \partial_m x_k \partial_n - x_k \partial_n x_l \partial_m$$

$$= x_l (\delta_{km} + x_k \partial_m) \partial_n - x_k (\delta_{nl} + x_l \partial_n) \partial_m$$

$$= \delta_{km} x_l \partial_n - \delta_{nl} x_k \partial_m$$

$$[X_i, X_j] = \varepsilon_{ilm} \varepsilon_{jkn} (\delta_{km} x_l \partial_n - \delta_{nl} x_k \partial_m)$$

$$= \varepsilon_{ilk} \varepsilon_{jkn} x_l \partial_n - \varepsilon_{ilm} \varepsilon_{jkl} x_k \partial_m$$

$$= \varepsilon_{iml} \varepsilon_{jkl} x_k \partial_m - \varepsilon_{ilk} \varepsilon_{jnk} x_l \partial_n$$

Property of ε-Tensor: $\varepsilon_{ijn} \varepsilon_{kln} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$ (sum over n!)

$$= (\delta_{ij} \delta_{mk} - \delta_{ik} \delta_{mj}) x_k \partial_m - (\delta_{ij} \delta_{ln} - \delta_{in} \delta_{lj}) x_l \partial_n$$

$$= \delta_{in} \delta_{jl} x_l \partial_n - \delta_{ik} \delta_{jm} x_k \partial_m$$

($n \to m$, $l \to k$, $l \to k$, $n \to m$)

$$= \underbrace{(\delta_{im} \delta_{jk} - \delta_{ik} \delta_{jm})}_{\varepsilon_{ijr} \varepsilon_{mkr}} x_k \partial_m$$

$$= \varepsilon_{ijr} \varepsilon_{mkr} x_k \partial_m = \varepsilon_{ijr} (-\varepsilon_{rkm} x_k \partial_m)$$

$$= \underline{\varepsilon_{ijr} X_r} \quad \rightsquigarrow \quad \underline{c_{ij}{}^r = \varepsilon_{ijr}}$$

<u>Cartan-Metric:</u>

$$g_{\kappa\ell} = C_{\kappa r}{}^{s} C_{\ell s}{}^{r}$$

$$= \varepsilon_{\kappa r s}\, \varepsilon_{\ell s r} = -\varepsilon_{\kappa r s}\, \varepsilon_{\ell r s}$$

$$= -(\delta_{\kappa\ell}\,\delta_{rr} - \delta_{\kappa r}\,\delta_{r\ell})$$

$$= -(3\,\delta_{\kappa\ell} - \delta_{\kappa\ell}) = -2\,\delta_{\kappa\ell}$$

$$\rightsquigarrow \quad g^{\kappa\ell} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad g_{\kappa\ell} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

<u>Casimir-Operator:</u> $C := g^{\kappa\ell} X_\kappa X_\ell$

$$= -\tfrac{1}{2}\vec{X}^2 = \frac{1}{2\hbar^2}\vec{L}^2$$

Problem 12: Generators of SO(4) on $L^2(\mathbb{R}^4)$

<u>General considerations:</u> Let $L_{\mu\nu} := x_\mu \partial_\nu - x_\nu \partial_\mu$

with $x_\mu \hat{=} (x_1, \dots, x_n) \in \mathbb{R}^n$

obviously $L_{\mu\nu}$ is generator of rotation in μ-ν-plane

$\rightsquigarrow$ set of all $L_{\mu\nu}$ generate rotations in $\mathbb{R}^n$ $\rightsquigarrow$

form SO(n)

Lets first construct the general algebra so(n) formed by $L_{\mu\nu}$'s

$$[L_{\mu\nu}, L_{\rho\sigma}] = (x_\mu \partial_\nu - x_\nu \partial_\mu)(x_\rho \partial_\sigma - x_\sigma \partial_\rho) - (x_\rho \partial_\sigma - x_\sigma \partial_\rho)(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

$$= x_\mu \partial_\nu x_\rho \partial_\sigma - x_\mu \partial_\nu x_\sigma \partial_\rho - x_\nu \partial_\mu x_\rho \partial_\sigma + x_\nu \partial_\mu x_\sigma \partial_\rho$$

$$- x_\rho \partial_\sigma x_\mu \partial_\nu + x_\rho \partial_\sigma x_\nu \partial_\mu + x_\sigma \partial_\rho x_\mu \partial_\nu - x_\sigma \partial_\rho x_\nu \partial_\mu$$

$$= x_\mu(\delta_{\nu\rho} + \overset{2}{x_\rho \partial_\nu})\partial_\sigma - x_\mu(\delta_{\nu\sigma} + \overset{4}{x_\sigma \partial_\nu})\partial_\rho - x_\nu(\delta_{\mu\rho} + \overset{3}{x_\rho \partial_\mu})\partial_\sigma + x_\nu(\delta_{\mu\sigma} + \overset{1}{x_\sigma \partial_\mu})\partial_\rho$$

$$- x_\rho(\delta_{\sigma\mu} + \overset{2}{x_\mu \partial_\sigma})\partial_\nu + x_\rho(\delta_{\sigma\nu} + \overset{3}{x_\nu \partial_\sigma})\partial_\mu + x_\sigma(\delta_{\rho\mu} + \overset{4}{x_\mu \partial_\rho})\partial_\nu - x_\sigma(\delta_{\rho\nu} + \overset{1}{x_\nu \partial_\rho})\partial_\mu$$

$$= \delta_{\nu\rho}(x_\mu \partial_\sigma - x_\sigma \partial_\mu) - \delta_{\nu\sigma}(x_\mu \partial_\rho - x_\rho \partial_\mu) - \delta_{\mu\rho}(x_\nu \partial_\sigma - x_\sigma \partial_\nu)$$

$$+ \delta_{\mu\sigma}(x_\nu \partial_\rho - x_\rho \partial_\nu)$$

$$= \delta_{\nu\rho} L_{\mu\sigma} - \delta_{\nu\sigma} L_{\mu\rho} - \delta_{\mu\rho} L_{\nu\sigma} + \delta_{\mu\sigma} L_{\nu\rho} \quad (*)$$

so(n)-algebra!

a)

Now $n=4$ and $(x_1, x_2, x_3, x_4) = (x, y, z, t) \in \mathbb{R}^4$

$M_1 = L_{32}$, $M_2 = L_{13}$, $M_3 = L_{21}$

$N_1 = L_{14}$, $N_2 = L_{24}$, $N_3 = L_{34}$ $\quad (\Rightarrow N_i = L_{i4})$

$\leadsto$ $[M_1, M_2] = [L_{32}, L_{13}] \overset{(*)}{=} L_{21} = M_3$

$[M_2, M_3] = [L_{13}, L_{21}] \overset{(*)}{=} L_{32} = M_1$

$[M_3, M_1] = [L_{21}, L_{32}] \overset{(*)}{=} L_{13} = M_2$

$\Rightarrow$ $[M_i, M_j] = \varepsilon_{ijk} M_k$

$\leadsto$ $[M_1, N_1] = [L_{32}, L_{14}] \overset{(*)}{=} 0$

$[M_2, N_2] = [L_{13}, L_{24}] \overset{(*)}{=} 0$

$[M_3, N_3] = [L_{21}, L_{34}] = 0$

$\Rightarrow$ $[M_i, N_i] = 0$

$$\left.\begin{aligned}
[M_1, N_2] &= [L_{32}, L_{24}] = L_{34} = N_3 \\
[M_2, N_3] &= [L_{13}, L_{34}] = L_{14} = N_1 \\
[M_3, N_1] &= [L_{21}, L_{14}] = L_{24} = N_2 \\
[M_1, N_3] &= [L_{32}, L_{34}] = -L_{24} = -N_2 \\
[M_2, N_1] &= [L_{13}, L_{14}] = -L_{34} = -N_3 \\
[M_3, N_2] &= [L_{21}, L_{24}] = -L_{14} = -N_1
\end{aligned}\right\} \underline{\underline{[M_i, N_j] = \varepsilon_{ijk} N_k}}$$

$$[N_i, N_j] = [L_{i4}, L_{j4}] = -L_{ij} = L_{ji} = \underline{\underline{\varepsilon_{ijk} M_k}}$$

b) $J_k := \frac{1}{2}(M_k + N_k)$

$$\rightsquigarrow 4[J_k, J_l] = [M_k + N_k, M_l + N_l]$$

$$= [M_k, M_l] + [M_k, N_l] + [N_k, M_l] + [N_k, N_l]$$

$$= \varepsilon_{klm} M_m + \varepsilon_{klm} N_m - \varepsilon_{lkm} N_m + \varepsilon_{klm} M_m$$

$$= \varepsilon_{klm}(2M_m + 2N_m) = 4\varepsilon_{klm} J_m$$

$\Rightarrow \underline{\underline{[J_k, J_l] = \varepsilon_{klm} J_m}}$ see Problem 1c) so(3) algebra

$K_i := \frac{1}{2}(M_i - N_i)$

$$\rightsquigarrow 4[K_i, K_j] = [M_i - N_i, M_j - N_j] = [M_i, M_j] - [M_i, N_j] - [N_i, M_j] + [N_i, N_j]$$

$$= \varepsilon_{ijk} M_k - \varepsilon_{ijk} N_k + \varepsilon_{jik} N_k + \varepsilon_{ijk} M_k$$

$$= 2\varepsilon_{ijk}(M_k - N_k) = 4\varepsilon_{ijk} K_k$$

$\Rightarrow [K_i, K_j] = \varepsilon_{ijk} K_k$ so(3)

$$[J_i, K_j] = [M_i + N_i, M_j - N_j] = [M_i, M_j] + [N_i, M_j] - [M_i, N_j] - [N_i, N_j]$$

$$= \varepsilon_{ijk} M_k - \varepsilon_{jik} N_k - \varepsilon_{ijk} N_k - \varepsilon_{ijk} M_k = \underline{\underline{0}}$$

decoupled! $\rightsquigarrow$ so(4) = so(3) $\oplus$ so(3)

Casimir op. of so(4) $\Longleftrightarrow$ 2 Casimir op. of so(3)

$\leadsto$ $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$ is Casimir of first so(3)

$\vec{K}^2 = K_1^2 + K_2^2 + K_3^2$ is Casimir of second so(3)

$$\Rightarrow \vec{J}^2 = \frac{1}{4}(\vec{M}+\vec{N})^2 = \frac{1}{4}(\vec{M}^2 + \vec{M}\cdot\vec{N} + \vec{N}\cdot\vec{M} + \vec{N}^2)$$

$\vec{M}\cdot\vec{N} = \vec{N}\cdot\vec{M}$ as $[M_i, N_i] = 0$

$$= \frac{1}{4}(\vec{M}^2 + 2\vec{M}\cdot\vec{N} + \vec{N}^2)$$

$$\vec{K}^2 = \frac{1}{4}(\vec{M}-\vec{N})^2 = \frac{1}{4}(\vec{M}^2 - 2\vec{M}\cdot\vec{N} + \vec{N}^2)$$

$$\leadsto \left.\begin{aligned} \vec{M}\cdot\vec{N} &= \vec{J}^2 - \vec{K}^2 \\ \frac{1}{2}(\vec{M}^2 + \vec{N}^2) &= \vec{J}^2 + \vec{K}^2 \end{aligned}\right\}$$

const for UIR of both so(3)

let (j, k) denote such UIR

$j, k = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$