4. Solution to Homework in "Group Theory for Physicists" **SoSe 25**

Problem 9: Characters of $SO(3)$ and $SU(2)$

a) Let D be a representation of $SO(3) \simeq SU(2)$ then

$$D(R(\vec{\omega})) = D(R\mathrm{e}^{-\mathrm{i}\varphi J_3})D(R\mathrm{e}^{-\mathrm{i}\theta J_2})D(R\mathrm{e}^{-\mathrm{i}\omega J_3})D(R\mathrm{e}^{\mathrm{i}\theta J_2})D(R\mathrm{e}^{\mathrm{i}\varphi J_3})$$

and

$$\chi(R(\vec{\omega})) = \mathrm{Tr}\, D(R(\vec{\omega})) = \mathrm{Tr}\, \left[D(R\mathrm{e}^{-\mathrm{i}\omega J_3})\right] = \chi(\omega) \quad \#$$

b)

$$\begin{aligned}
\chi^j(\omega) &= \mathrm{Tr}\, D^j(R) = \sum_{m=-j}^{j} \langle jm|\mathrm{e}^{-\mathrm{i}\omega J_3}|jm\rangle \\
&= \sum_{m=-j}^{j} \mathrm{e}^{-\mathrm{i}\omega m} = \mathrm{e}^{-\mathrm{i}\omega j} \sum_{k=0}^{2j+1} (\mathrm{e}^{\mathrm{i}\omega})^k \qquad \text{geom. series} \\
&= \mathrm{e}^{-\mathrm{i}\omega j} \frac{1-\mathrm{e}^{\mathrm{i}\omega(2j+1)}}{1-\mathrm{e}^{\mathrm{i}\omega}} = \frac{\mathrm{e}^{-\mathrm{i}\omega(j+\frac{1}{2})} - \mathrm{e}^{\mathrm{i}\omega(j+\frac{1}{2})}}{\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega} - \mathrm{e}^{\frac{\mathrm{i}}{2}\omega}} \\
&= \frac{\sin(j+\frac{1}{2})\omega}{\sin\frac{\omega}{2}} \quad \#
\end{aligned}$$

c)

$$\begin{aligned}
\chi^{j_1}(\omega)\chi^{j_2}(\omega) &= \frac{\mathrm{e}^{-\mathrm{i}\omega(j_1+\frac{1}{2})} - \mathrm{e}^{\mathrm{i}\omega(j_1+\frac{1}{2})}}{2\mathrm{i}\sin\frac{\omega}{2}} \sum_{m=-j_2}^{j_2} \mathrm{e}^{\mathrm{i}\omega m} \\
& \quad \text{let} \quad j_1 \geq j_2 \quad \text{without loss of generlity} \\
&= \frac{1}{2\mathrm{i}\sin\frac{\omega}{2}} \sum_{m=-j_2}^{j_2} \left(\mathrm{e}^{\mathrm{i}\omega(j_1+m+\frac{1}{2})} - \mathrm{e}^{-\mathrm{i}\omega(j_1+m+\frac{1}{2})}\right) \\
& \quad \text{set} \quad j = j_1 + m \\
&= \frac{1}{2\mathrm{i}\sin\frac{\omega}{2}} \sum_{j=j_1-j_2}^{j_1+j_2} \left(\mathrm{e}^{\mathrm{i}\omega(j+\frac{1}{2})} - \mathrm{e}^{-\mathrm{i}\omega(j+\frac{1}{2})}\right) \\
&= \sum_{j=j_1-j_2}^{j_1+j_2} \frac{\sin(j+\frac{1}{2})\omega}{\sin\frac{\omega}{2}} = \sum_{j=|j_1-j_2|}^{j_1+j_2} \chi^j(\omega) \\
& \quad \text{as LHS is symmetric in} \quad j_1 \leftrightarrow j_2
\end{aligned}$$

Hence the addition of two angular momenta representation spaces is given by the reduction

$$D^{j_1} \otimes D^{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D^j$$

Problem 10: Generators of $SO(3)$ for $j = 1$ (3-dim. representation)

Explicit calculations result in

$$L_1 L_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L_2 L_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L_2 L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad L_3 L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L_3 L_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L_1 L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$L_1^2 = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right), \qquad L_2^2 = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \qquad L_3^2 = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right),$$

a)

$$\left.\begin{aligned} [L_1, L_2] &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{i} L_3 \\ [L_2, L_3] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \mathrm{i} L_1 \\ [L_3, L_1] &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathrm{i} L_2 \end{aligned}\right\} \qquad [L_k, L_m] = \mathrm{i}\varepsilon_{kmn} L_n$$

b) L_k^2 is unit matrix in subspace orthogonal to the k-axis and vanishes on the one-dim. subspace spanned by the k-axis

Hence its matrix elements are given by

$$\left(L_k^2\right)_{mn} = \delta_{mn} - \delta_{km}\delta_{kn}$$

Obviously

$$L_k^0 = 1 \qquad \text{and} \qquad L_k^{2m} = L_k^2 \qquad \text{for} \qquad m = 1, 2, 3, \ldots$$

and

$$L_k^{2m+1} = L_k \qquad \text{for} \qquad m = 0, 1, 2, 3, \ldots$$

c)

$$\begin{aligned}
\exp\{-\mathrm{i}\varphi L_k\} &= \sum_{n=0}^{\infty} \frac{1}{n!}\,(-\mathrm{i}\varphi L_k)^n = \sum_{m=0}^{\infty} \frac{1}{(2m)!}\,(-\mathrm{i}\varphi L_k)^{2m} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}\,(-\mathrm{i}\varphi L_k)^{2m+1} \\
&= \underbrace{1 - L_k^2}_{m=0 \text{ term}} + L_k^2 \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!}\varphi^{2m}}_{\cos\varphi} + (-\mathrm{i}L_k) \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\varphi^{2m+1}}_{\sin\varphi} \\
&= 1 + L_k^2(\cos\varphi - 1) - \mathrm{i}L_k \sin\varphi \qquad \#
\end{aligned}$$

Explicitly

$$\exp\{-\mathrm{i}\varphi L_1\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & 0 \\ 0 & 0 & \cos\varphi \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sin\varphi \\ 0 & -\sin\varphi & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

$$\exp\{-\mathrm{i}\varphi L_2\} = \begin{pmatrix} \cos\varphi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\varphi \end{pmatrix} - \begin{pmatrix} 0 & 0 & -\sin\varphi \\ 0 & 0 & 0 \\ \sin\varphi & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

$$\exp\{-\mathrm{i}\varphi L_3\} = \begin{pmatrix} \cos\varphi & 0 & 0 \\ 0 & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \sin\varphi & 0 \\ -\sin\varphi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{Tr}\,\left[\mathrm{e}^{-\mathrm{i}\varphi L_k}\right] = \mathrm{Tr}\,\left[1 + L_k^2(\cos\varphi - 1) - \mathrm{i}L_k\sin\varphi\right] = 3 + 2(\cos\varphi - 1) = 1 + 2\cos\varphi$$

$$\chi^1(\varphi) = \frac{\sin\frac{3}{2}\varphi}{\sin\frac{\varphi}{2}} = \frac{3\sin\frac{\varphi}{2} - 4\sin^2\frac{\varphi}{2}}{\sin\frac{\varphi}{2}} = 3 - 4\sin^2\tfrac{\varphi}{2} = 1 + 2(1 - \sin^2\tfrac{\varphi}{2}) = 1 + 2\cos\varphi \quad \#$$

Problem 11: Generators of $SO(3)$ in $L^2(\mathbb{R}^3)$

a) Use result of problem 10c)

$$\exp\{-\mathrm{i}\delta\alpha^a L_a\} = 1 + L_a^2(\cos\delta\alpha^a - 1) - \mathrm{i}L_a \sin\delta\alpha^a \approx 1 - \mathrm{i}L_a\delta\alpha^a + O((\delta\alpha)^2)$$

Hence

$$g(\vec{\delta\alpha}) \approx 1 - \mathrm{i}\sum_{a=1}^{3}\delta\alpha^a L_a = 1 + \begin{pmatrix} 0 & -\delta\alpha^3 & \delta\alpha^2 \\ \delta\alpha^3 & 0 & -\delta\alpha^1 \\ -\delta\alpha^2 & \delta\alpha^1 & 0 \end{pmatrix}$$

b)

$$\delta\vec{x} = \vec{x}\,' - \vec{x} = \begin{pmatrix} 0 & -\delta\alpha^3 & \delta\alpha^2 \\ \delta\alpha^3 & 0 & -\delta\alpha^1 \\ -\delta\alpha^2 & \delta\alpha^1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2\delta\alpha^3 + x_3\delta\alpha^2 \\ x_1\delta\alpha^3 - x_3\delta\alpha^1 \\ -x_1\delta\alpha^2 + x_2\delta\alpha^1 \end{pmatrix} =: \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix}$$

Hence

$$\left.\begin{aligned} X_1 &= -\sum_{k=1}^{3}\frac{\delta x_k}{\delta\alpha_1}\,\partial_k = x_3\partial_2 - x_2\partial_3 \\ X_2 &= -\sum_{k=1}^{3}\frac{\delta x_k}{\delta\alpha_2}\,\partial_k = x_1\partial_3 - x_3\partial_1 \\ X_3 &= -\sum_{k=1}^{3}\frac{\delta x_k}{\delta\alpha_3}\,\partial_k = x_2\partial_1 - x_1\partial_2 \end{aligned}\right\} \quad \vec{X} = -\vec{x}\times\vec{\nabla} = \frac{1}{\mathrm{i}\hbar}\,\vec{x}\times\vec{p} = \frac{1}{\mathrm{i}\hbar}\,\vec{L}$$

c) As $X_i = -\varepsilon_{ijk}x_j\partial_k$ (sum over repeated indices) $\quad\Longrightarrow$

$$[X_i, X_j] = \varepsilon_{ilm}\varepsilon_{jkn}[x_l\partial_m, x_k\partial_n]$$

Consider

$$\begin{aligned} [x_l\partial_m, x_k\partial_n] &= x_l\partial_m x_k\partial_n - x_k\partial_n x_l\partial_m \\ &= x_l\left(\delta_{km} + \underline{x_k\partial_m}\right)\partial_n - x_k\left(\delta_{nl} + \underline{x_l\partial_n}\right)\partial_m \\ &= \delta_{km}x_l\partial_n - \delta_{nl}x_k\partial_m \end{aligned}$$

Hence

$$
\begin{aligned}
[X_i, X_j] &= \varepsilon_{ilm}\varepsilon_{jkn}\left(\delta_{km}x_l\partial_n - \delta_{nl}x_k\partial_m\right)\\
&= \varepsilon_{ilk}\varepsilon_{jkn}x_l\partial_n - \varepsilon_{ilm}\varepsilon_{jkl}x_k\partial_m\\
&\quad \text{use} \quad \varepsilon_{jkn} = -\varepsilon_{jnk} \quad \text{and} \quad \varepsilon_{ilm} = -\varepsilon_{iml}\\
&= \varepsilon_{iml}\varepsilon_{jkl}x_k\partial_m - \varepsilon_{ilk}\varepsilon_{jnk}x_l\partial_n\\
&\quad \text{use} \quad \varepsilon_{rst}\varepsilon_{uvt} = \left(\delta_{ru}\delta_{sv} - \delta_{rv}\delta_{su}\right)\\
&= \left(\underline{\delta_{ij}\delta_{mk}} - \delta_{ik}\delta_{mj}\right)x_k\partial_m - \left(\underline{\delta_{ij}\delta_{ln}} - \delta_{in}\delta_{lj}\right)x_l\partial_n\\
&= \delta_{in}\delta_{lj}x_l\partial_n - \delta_{ik}\delta_{mj}x_k\partial_m\\
&\quad \text{replace in 1. term} \quad n \to m\,, \quad l \to k\\
&= \underbrace{\left(\delta_{im}\delta_{jk} - \delta_{ik}\delta_{jm}\right)}_{\varepsilon_{ijr}\varepsilon_{mkr}}x_k\partial_m\\
&= -\varepsilon_{ijr}\varepsilon_{rkm}x_k\partial_m = \varepsilon_{ijr}X_r \qquad \Longrightarrow \qquad \boxed{{c_{ij}}^r = \varepsilon_{ijr}} \quad \#
\end{aligned}
$$

Cartan metric:

$$
\begin{aligned}
g_{kl} :=&\ {c_{kr}}^s{c_{ls}}^r = \varepsilon_{krs}\varepsilon_{lsr} = -\varepsilon_{krs}\varepsilon_{lrs}\\
=&\ -\left(\delta_{kl}\delta_{rr} - \delta_{kr}\delta_{rl}\right)\\
=&\ -\left(3\delta_{kl} - \delta_{kl}\right) = -2\delta_{kl} \qquad \#
\end{aligned}
$$

Metric tensors:

$$
(g_{kl}) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad\qquad \left(g^{kl}\right) = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}
$$

Casimir operator:

$$
C := g^{kl}X_kX_l = -\tfrac{1}{2}\,\vec{X}^{\,2} = \tfrac{1}{2\hbar}\,\vec{L}^{\,2}
$$

Problem 12: Generators of $SO(4)$ in $L^2(\mathbb{R}^4)$

General considerations:

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, x_μ and ∂_μ denote coordinate and derivative, respectively. Consider operator

$$L_{\mu\nu} := x_\mu \partial_\nu - x_\nu \partial_\mu$$

Obviously, $L_{\mu\nu}$ is generator of rotation in $x_\mu - x_\nu$-plane

$\Longrightarrow$ set of all $L_{\mu\nu} = -L_{\nu\mu}$ generates rotations in $\mathbb{R}^n$

There exist $n(n-1)/2$ independent generators of $SO(n)$.

Derivation of the $so(n)$ algebra:

$$\begin{aligned}
[L_{\mu\nu}, L_{\rho\sigma}] = & \left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)\left(x_\rho\partial_\sigma - x_\sigma\partial_\rho\right) - \left(x_\rho\partial_\sigma - x_\sigma\partial_\rho\right)\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \\
= & \; x_\mu\partial_\nu x_\rho\partial_\sigma - x_\mu\partial_\nu x_\sigma\partial_\rho - x_\nu\partial_\mu x_\rho\partial_\sigma + x_\mu\partial_\nu x_\sigma\partial_\rho \\
& -x_\rho\partial_\sigma x_\mu\partial_\nu + x_\rho\partial_\sigma x_\nu\partial_\mu + x_\sigma\partial_\rho x_\mu\partial_\nu - x_\sigma\partial_\rho x_\nu\partial_\mu \\
= & \; x_\mu\left(\delta_{\nu\rho} + \underline{x_\rho\partial_\nu}\right)\partial_\sigma - x_\mu\left(\delta_{\nu\sigma} + \underline{x_\sigma\partial_\nu}\right)\partial_\rho - x_\nu\left(\delta_{\mu\rho} + \underline{x_\rho\partial_\mu}\right)\partial_\sigma + x_\nu\left(\delta_{\mu\sigma} + \underline{x_\sigma\partial_\mu}\right)\partial_\rho \\
& -x_\rho\left(\delta_{\mu\sigma} + \underline{x_\mu\partial_\sigma}\right)\partial_\nu + x_\rho\left(\delta_{\nu\sigma} + \underline{x_\nu\partial_\sigma}\right)\partial_\mu + x_\sigma\left(\delta_{\mu\rho} + \underline{x_\mu\partial_\rho}\right)\partial_\nu - x_\sigma\left(\delta_{\nu\rho} + \underline{x_\nu\partial_\rho}\right)\partial_\mu \\
= & \; \delta_{\nu\rho}\left(x_\mu\partial_\sigma - x_\sigma\partial_\mu\right) - \delta_{\nu\sigma}\left(x_\mu\partial_\rho - x_\rho\partial_\mu\right) - \delta_{\mu\rho}\left(x_\nu\partial_\sigma - x_\sigma\partial_\nu\right) + \delta_{\mu\sigma}\left(x_\nu\partial_\rho - x_\rho\partial_\nu\right) \\
= & \; \delta_{\nu\rho}L_{\mu\sigma} - \delta_{\nu\sigma}L_{\mu\rho} - \delta_{\mu\rho}L_{\nu\sigma} + \delta_{\mu\sigma}L_{\nu\rho}
\end{aligned}$$

a) Now $n = 4$ and $(x_1, x_2, x_3, x_4) = (x, y, z, t) \in \mathbb{R}^4$

We identify

$$\begin{array}{lll}
M_1 = L_{32} & M_2 = L_{13} & M_3 = L_{21} \\
N_1 = L_{14} & N_2 = L_{24} & N_3 = L_{34}
\end{array}$$

Now we can use above result of $so(n)$ algebra:

$$\left.\begin{aligned}
[M_1, M_2] &= [L_{32}, L_{13}] = L_{21} = M_3 \\
[M_2, M_3] &= [L_{13}, L_{21}] = L_{32} = M_1 \\
[M_3, M_1] &= [L_{21}, L_{32}] = L_{13} = M_2
\end{aligned}\right\} \qquad [M_i, M_j] = \varepsilon_{ijk} M_k$$

$$\left.\begin{aligned}
[M_1, N_1] &= [L_{32}, L_{14}] = 0 \\
[M_2, N_2] &= [L_{13}, L_{24}] = 0 \\
[M_3, N_2] &= [L_{21}, L_{34}] = 0
\end{aligned}\right\} \qquad [M_i, N_i] = 0$$

$$\left.\begin{aligned}
[M_1, N_2] &= [L_{32}, L_{24}] = L_{34} = N_3 \\
[M_2, N_3] &= [L_{13}, L_{34}] = L_{14} = N_1 \\
[M_3, N_1] &= [L_{21}, L_{14}] = L_{24} = N_2 \\
[M_1, N_3] &= [L_{32}, L_{34}] = L_{24} = -N_3 \\
[M_2, N_1] &= [L_{13}, L_{14}] = L_{34} = -N_1 \\
[M_3, N_2] &= [L_{21}, L_{24}] = L_{14} = -N_2
\end{aligned}\right\} \qquad [M_i, N_j] = \varepsilon_{ijk} N_k$$

$$[N_i, N_j] = [L_{i4}, L_{j4}] = -L_{ij} = L_{ji} = \varepsilon_{ijk} M_k$$

b) Let $J_k := \frac{1}{2}(M_k + N_k)$

$$\begin{aligned}
4[J_k, J_l] &= [M_k + N_k, M_l + N_l] \\
&= [M_k, M_l] + [M_k, N_l] + [N_k, M_l] + [N_k, N_l] \\
&= \varepsilon_{klm} M_m + \varepsilon_{klm} N_m - \varepsilon_{lkm} N_m + \varepsilon_{klm} M_m \\
&= \varepsilon_{klm}(2M_m + 2N_m) = 4\varepsilon_{klm} J_m
\end{aligned}$$

$$\Longrightarrow \qquad [J_k, J_l] = \varepsilon_{klm} J_m \qquad so(3) - \text{algebra}$$

Let $K_i := \frac{1}{2}(M_i - N_i)$

$$\begin{aligned}
4[K_i, K_j] &= [M_i - N_i, M_j - N_j] \\
&= [M_i, M_j] - [M_i, N_j] - [N_i, M_j] + [N_i, N_j] \\
&= \varepsilon_{ijk} M_k - \varepsilon_{ijk} N_k + \varepsilon_{jik} N_k + \varepsilon_{ijk} M_k \\
&= 2\varepsilon_{ijk}(M_k - N_k) = 4\varepsilon_{ijk} K_k
\end{aligned}$$

$$\Longrightarrow \qquad [K_i, K_j] = \varepsilon_{ijk} K_k \qquad so(3) - \text{algebra}$$

Finally we show the decoupling of both subalgebras

$$\begin{aligned}
[J_i, K_j] &= [M_i + N_i, M_j - N_j] \\
&= [M_i, M_j] - [M_i, N_j] + [N_i, M_j] - [N_i, N_j] \\
&= \varepsilon_{ijk} M_k - \varepsilon_{ijk} N_k - \varepsilon_{jik} N_k - \varepsilon_{ijk} M_k \\
&= 0
\end{aligned}$$

$$\Longrightarrow \qquad so(4) = so(3) \oplus so(3)$$

Casimir operators of $so(4)$

We have two Casimir from both $so(3)$ algebras:

$$\vec{J}^2 = J_1^2 + J_2^2 + J_3^2\,, \qquad \vec{K}^{\,2} = K_1^2 + K_2^2 + K_3^2$$

The corresponding labels of their UIR are $j, k = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

Hence, the UIR of $so(4)$ can be denoted by the pair (j, k)

Some relations

$$\begin{aligned}
\vec{J}^2 =&\ \tfrac{1}{4}(\vec{M} + \vec{N})^2 = \tfrac{1}{4}(\vec{M}^{\,2} + \vec{M}\cdot\vec{N} + \vec{N}\cdot\vec{M} + \vec{N}^{\,2}) \\
&\quad \text{but} \quad \vec{M}\cdot\vec{N} = \vec{N}\cdot\vec{M} \quad \text{as} \quad [M_i, N_i] = 0 \\
=&\ \tfrac{1}{4}(\vec{M}^{\,2} + 2\vec{M}\cdot\vec{N} + \vec{N}^{\,2})
\end{aligned}$$

$$\vec{K}^{\,2} = \tfrac{1}{4}(\vec{M} - \vec{N})^2 = \tfrac{1}{4}(\vec{M}^{\,2} - 2\vec{M}\cdot\vec{N} + \vec{N}^{\,2})$$

$$\vec{J}^2 - \vec{K}^{\,2} = \vec{M}\cdot\vec{N}$$

$$\vec{J}^2 + \vec{K}^{\,2} = \tfrac{1}{4}(\vec{M}^{\,2} + \vec{N}^{\,2})$$