3. Solution to Homework in "Group Theory for Physicists" SoSe 25

Problem 6: The classes of D_n

Generators d, s obey relations $d^n = e = s^2$ and $sd = d^{-1}s$

a) Classes are defined by one group element g via $\{g_1 g g_1^{-1}, \ldots, g_n g g_n^{-1}\}$, where in essence we need to look into the generators only.

Hence we have below classes of pure rotations:

- $\{e\}$ neutral element is always a class by its own.
- $\{d, d^{n-1}\}$ due to $sds^{-1} = d^{-1} = d^{n-1}$
- $\{d^2, d^{n-2}\}$ due to $sd^2s^{-1} = d^{-2} = d^{n-2}$
- $\vdots$
- $\{d^k, d^{n-k}\}$ due to $sd^ks^{-1} = d^{-k} = d^{n-k}$
- $\vdots$
- Ends for even n with $\{d^{\frac{n}{2}}\}$ or for odd n with $\{d^{\frac{n-1}{2}}, d^{\frac{n+1}{2}}\}$

Now those with reflection.

- For even n we have two classes $\{sd, sd^3, \ldots, sd^{n-1}\}$ and $\{sd^2, sd^4, \ldots, sd^n\}$
- For odd n only one class exists $\{sd, sd^2, \ldots, sd^n\}$

This follows from the relations

$$d^\ell (sd^{2k}) d^{-\ell} = sd^{2(k-\ell)}\,, \qquad s(sd^{2k})s^{-1} = d^{2k}s = sd^{n-2k}$$

and

$$d^\ell (sd^{2k+1}) d^{-\ell} = sd^{2(k-\ell)+1}\,, \qquad s(sd^{2k+1})s^{-1} = sd^{n-(2k+1)}$$

Result:

n even: $1 + \frac{n}{2} + 2 = 4 + \left(\frac{n}{2} - 1\right)$ classes; 4 1-d and $\left(\frac{n}{2} - 1\right)$ 2-d UIR

n odd: $1 + \frac{n-1}{2} + 1 = 2 + \frac{n-1}{2}$ classes; 2 1-d and $\frac{n-1}{2}$ 2-d UIR

b) Character table of D_4

For the four 1-dim. UIR we have $\chi^{(1,r)}(g) = D^{(1,r)}(g)$, $r = 1, 2, 3, 4$.

There is only one 2-dim. UIR given by the generators

$$D^{(2,1)}(d) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\pi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} \end{pmatrix} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad D^{(2,1)}(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Hence

$$\chi^{(2,1)}(d) = \mathrm{Tr} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = 0$$

$$\chi^{(2,1)}(d^2) = \mathrm{Tr} \left[\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \right] = \mathrm{Tr} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -2$$

and rest is trivial.

Character table:

D_4	$\{e\}$	$\{d, d^3\}$	$\{d^2\}$	$\{sd, sd^3\}$	$\{s, sd^2\}$
$D^{(1,1)}$	1	1	1	1	1
$D^{(1,2)}$	1	1	1	-1	-1
$D^{(1,3)}$	1	-1	1	-1	1
$D^{(1,4)}$	1	-1	1	1	-1
$D^{(2,1)}$	2	0	-2	0	0

Problem 7: Projection operators

Definition for finite group G of order n and UIRs D^j with dimension d_j:

$$\mathbb{E}^j = \frac{d_j}{n} \sum_{g \in G} \chi^{j\,*}(g) D(g)\,,$$

with D being unitary but reducible.

a) Hermicity:

$$\mathbb{E}^{j\,\dagger} = \frac{d_j}{n} \sum_{g \in G} \chi^j(g) D^*(g) = \frac{d_j}{n} \sum_{g \in G} \chi^j(g) D(g^{-1}) = \frac{d_j}{n} \sum_{g \in G} \chi^{j\,*}(g) D(g) = \mathbb{E}^j$$

b) Orthonormality:

$$\begin{aligned}
\mathbb{E}^j \mathbb{E}^k &= \frac{d_j d_k}{n^2} \sum_{g,g' \in G} \chi^j(g^{-1}) \chi^{k\,*}(g') D(g) D(g') \\
&\qquad \tilde{g} := g g' \quad \implies \quad g = \tilde{g}(g')^{-1} \quad \implies \quad g^{-1} = g'(\tilde{g})^{-1} \\
&= \frac{d_j d_k}{n^2} \sum_{\tilde{g},g' \in G} \underbrace{\chi^j(g'\tilde{g}^{-1})}_{D^j_{st}(g') D^j_{ts}(\tilde{g}^{-1})} \underbrace{\chi^{k\,*}(g')}_{D^{k\,*}_{rr}(g')} \underbrace{D(\tilde{g} g'^{-1}) D(g')}_{D(\tilde{g})} \\
&= \frac{d_j d_k}{n^2} \underbrace{\sum_{g' \in G} D^{k\,*}_{rr}(g') D^j_{st}(g')}_{\frac{n}{d_j} \delta_{sr} \delta_{tr} \delta_{jk}} \sum_{\tilde{g} \in G} D^{j\,*}_{st}(\tilde{g}) D(\tilde{g}) \\
&= \delta_{jk} \frac{d_j}{n} \sum_{\tilde{g} \in G} D^{j\,*}_{rr}(\tilde{g}) D(\tilde{g}) \\
&= \mathbb{E}^j \delta_{jk}
\end{aligned}$$

c) Completeness: Follows from a) and b) and D being fully reducible:

$$\sum_{\{j\}} \mathbb{E}^j = 1 \qquad \text{on reps. space of } D$$

Some details:

Let $D(g) = \sum_{\{k\}} c_k\, D^k(g)$, c_k multiplicity of UIR D^k in D, $\{k\}$ set of all UIR in D.

Let $m_r, n_r = 1, 2, \ldots, d_k$ basis labels of r-th occurrence of D^k in D, $r = 1, 2, \ldots, c_k$.

Then

$$\left(\mathbb{E}^j\right)_{m_r n_r} = \frac{d_j}{n} \sum_{g \in G} \chi^{j\,*}(g) D_{m_r n_r}(g) = \frac{d_j}{n} \sum_{g \in G} D^{j\,*}_{tt}(g) D_{m_r n_r}(g) = \delta_{jk} \delta_{m_r n_r}$$

- $c_j = 0$: UIR D^j is not in above decomposition, i.e., for all k we have $j \neq k$

  $\implies \qquad \mathbb{E}^j = 0$

- $\exists k = j$: For the restricted projector on any of the c_k subspace D^k follows $\mathbb{E}^j\big|_{D^k} = \mathbf{1}_{d_k}$

  $\implies \qquad \mathbb{E}^j$ projects onto $\Sigma^j = c_j\, D^j$ if $c_j \geq 1$

Problem 8: The dynamical $SO(4)$-symmetry of the Kepler-Problem

Kepler problem:

$$\frac{m}{2}\dot{\vec{r}}^2 - G\frac{Mm}{r} = E \qquad (N)$$

Reduced mass: $m := \frac{m_P M_S}{m_P + M_S}$, with m_P mass of planet and M_S mass of sun

Total mass: $M := M_S + m_P$

Gravitational pot.: $V(r) = -G\frac{M_S m_P}{r} = -G\frac{Mm}{r}$, where r is distance between both masses

a) New time s such that $\mathrm{d}s = \frac{1}{r}\mathrm{d}t$ and $\frac{\mathrm{d}t}{\mathrm{d}s} = t' = r \qquad \Longrightarrow$

$$\dot{\vec{r}} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\vec{r}'}{t'} = \frac{\vec{r}'}{r}$$

Inserting into (N) results in

$$\begin{aligned}
\frac{m}{2}\frac{\vec{r}'^2}{r^2} - G\frac{Mm}{r} &= E \qquad \Big| \qquad \times r^2 = t'^2 \\
\frac{m}{2}\vec{r}'^2 - GMmt' &= Et'^2 \\
\vec{r}'^2 - 2GMt' &= \frac{2E}{m}t'^2 \qquad (1)
\end{aligned}$$

b) Let $\quad r_E := GM/v_E^2 \quad$ with $\quad v_E^2 := \frac{2|E|}{m} \quad$ and $\quad E < 0$.

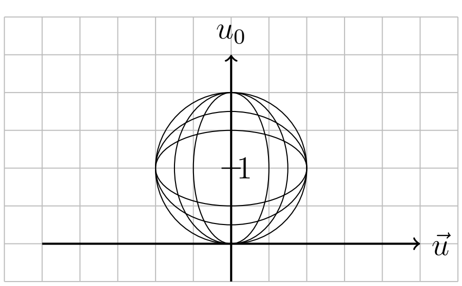
$\Longrightarrow \qquad \frac{2E}{m} = -v_E^2 \qquad$ insert this into (1) $\qquad \Longrightarrow$

$$\begin{aligned}
\vec{r}'^2 + v_E^2\left(t'^2 - 2r_E t'\right) &= 0 \\
\vec{r}'^2 + v_E^2\left(t'^2 - 2r_E t' + r_E^2\right) &= v_E^2 r_E^2 \\
\left(\frac{\vec{r}'}{v_E r_E}\right)^2 + \left(\frac{t'}{r_E} - 1\right)^2 &= 1
\end{aligned}$$

Let $\quad u_0 := t'/r_E \quad$ and $\quad \vec{u} := \vec{r}'/v_E r_E \quad$ then

$$\boxed{(u_0 - 1)^2 + \vec{u}^2 = 1}$$

The orbit of the 4-velocity $u = (u_0, \vec{u}^2)$ is given by a unit circle in $\mathbb{R}^4$.



Visualisation of the $SO(4)$ symmetry

c) Let $\quad E = 0 \quad$ in $\quad (1) \quad \Longrightarrow \quad \vec{r}'^2 = 2GMt'$

Let $\quad r_s := 2GM/c^2$, which (for c being the speed of light)

is in essence the Schwarzschild radius of the mass M.

Now we set $\quad u_0 := t'/r_s \quad$ and $\quad \vec{u} := \vec{r}'/r_s c \quad$ then

$$\boxed{\vec{u}^{\,2} = u_0}$$

The orbit of the 4-velocity $u = (u_0, \vec{u}^{\,2})$ moves on a paraboloid in $\mathbb{R}^4$.