

Problem 1: Cayley's theorem for $Z_2 = \{-1, 1\}$

Let $f(\pm 1) \in \mathbb{C}$ with $|f(\pm 1)| < \infty$ be well defined. Then for any $\sigma_0 \in Z_2$ with $\circ$ being the usual multiplication of numbers

$$\sum_{\sigma \in Z_2} f(\sigma \sigma_0) = f(-\sigma_0) + f(\sigma_0) = f(-1) + f(1) = \sum_{\sigma \in Z_2} f(\sigma).$$

Hence Cayley's rearrangement theorem is valid, that is,

$$\sum_{\sigma \in Z_2} f(\sigma \sigma_0) = \sum_{\sigma \in Z_2} f(\sigma), \quad \forall \sigma_0 \in Z_2.$$

Special cases:

- $f(\sigma) = \sigma$

$$\sum_{\sigma \in Z_2} \sigma = -1 + 1 = 0$$

- $f(\sigma) = \sigma^2$

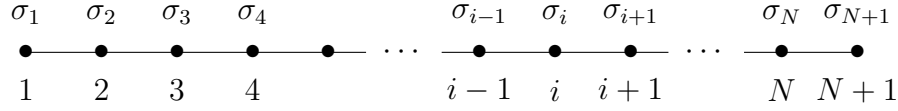
$$\sum_{\sigma \in Z_2} \sigma^2 = \sum_{\sigma \in Z_2} 1 = 2$$

- For Ising model

$$\begin{aligned} \cosh(\alpha) + \sigma \sinh(\alpha) &= \frac{1}{2} (e^\alpha + e^{-\alpha}) + \frac{\sigma}{2} (e^\alpha - e^{-\alpha}) \\ &= \frac{1+\sigma}{2} e^\alpha + \frac{1-\sigma}{2} e^{-\alpha} \\ &= \begin{cases} e^\alpha & \text{for } \sigma = 1 \\ e^{-\alpha} & \text{for } \sigma = -1 \end{cases} = e^{\alpha\sigma} \end{aligned}$$

Problem 2: The one-dimensional Ising model

One-dimensional lattice with $N + 1$ Ising-”Spins” attached, $\sigma_i \in Z_2$, $i = 1, 2, 3, \dots, N + 1$:



Hamiltonian:

$$H := -J \sum_{i=1}^N \sigma_i \sigma_{i+1}, \quad J \in \mathbb{R}.$$

Consider:

$$e^{-\beta H} = \exp \left\{ \beta J \sum_{i=1}^N \sigma_i \sigma_{i+1} \right\} = \prod_{i=1}^N e^{\beta J \sigma_i \sigma_{i+1}} = \prod_{i=1}^N \left(\cosh(\beta J) + \sigma_i \sigma_{i+1} \sinh(\beta J) \right)$$

Partition function:

$$\begin{aligned} Z(\beta) &= \sum_{\sigma_1 \in Z_2} \cdots \sum_{\sigma_{N+1} \in Z_2} \prod_{i=1}^N \cosh(\beta J) \left(1 + \lambda(\beta) \sigma_i \sigma_{i+1} \right) \quad \text{with} \quad \lambda(\beta) := \tanh(\beta J) \\ &= \cosh^N(\beta J) \left[\sum_{\sigma_1 \in Z_2} \cdots \sum_{\sigma_{N+1} \in Z_2} \left(1 + \lambda \sigma_1 \sigma_2 \right) \left(1 + \lambda \sigma_2 \sigma_3 \right) \cdots \left(1 + \lambda \sigma_N \sigma_{N+1} \right) \right] \end{aligned}$$

$$\begin{aligned} 1. \text{ sum : } & \sum_{\sigma_1 \in Z_2} \left(1 + \lambda \sigma_1 \sigma_2 \right) = 2 \\ 2. \text{ sum : } & \sum_{\sigma_2 \in Z_2} \left(1 + \lambda \sigma_2 \sigma_3 \right) = 2 \\ & \vdots \\ N\text{-th sum : } & \sum_{\sigma_N \in Z_2} \left(1 + \lambda \sigma_N \sigma_{N+1} \right) = 2 \\ (N+1)\text{-th sum : } & \sum_{\sigma_{N+1} \in Z_2} 1 = 2 \end{aligned}$$

Partition function for free boundary conditions:

$$\boxed{Z(\beta) = 2^{N+1} \cosh^N(\beta J)}$$

Free energy in thermodynamic limit:

$$F(\beta) := -\frac{1}{\beta} \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z(\beta) = -\frac{1}{\beta} \ln \left(2 \cosh \beta J \right)$$

Alternative way of doing the sum

Consider:

$$\begin{aligned}
& \sum_{\sigma_i \in Z_2} \left(1 + \lambda \sigma_{i-1} \sigma_i\right) \left(1 + \lambda \sigma_i \sigma_{i+1}\right) = \\
& = \sum_{\sigma_i \in Z_2} \left(1 + \lambda \sigma_{i-1} \sigma_i + \lambda \sigma_i \sigma_{i+1} + \lambda^2 \sigma_{i-1} \sigma_i^2 \sigma_{i+1}\right) \\
& = 2 + 0 + 0 + 2\lambda^2 \sigma_{i-1} \sigma_{i+1} \\
& = 2 \left(1 + \lambda^2 \sigma_{i-1} \sigma_{i+1}\right)
\end{aligned}$$

Symbolically:

$$\frac{1}{2} \sum_{\sigma_i \in Z_2} \begin{array}{c} \lambda \quad \lambda \\ \bullet \text{---} \bullet \text{---} \bullet \\ i-1 \quad i \quad i+1 \end{array} = \begin{array}{c} \lambda^2 \\ \bullet \text{---} \circ \text{---} \bullet \\ i-1 \quad \quad i+1 \end{array}$$

Doing the $N - 1$ intermediate sums gives

$$Z(\beta) = 2^{N-1} \cosh^N(\beta J) \sum_{\sigma_1 \in Z_2} \sum_{\sigma_{N+1} \in Z_2} \left(1 + \lambda^N \sigma_1 \sigma_{N+1}\right)$$

For open boundary condition we recover the result on pervious page

For periodic boundary conditions we have $\sigma_{N+1} = \sigma_1$

$$Z_{PB}(\beta) = 2^{N-1} \cosh^N(\beta J) \sum_{\sigma_1 \in Z_2} \left(1 + \lambda^N\right) = 2^N \cosh^N(\beta J) \left(1 + \lambda^N\right)$$

and

$$F_{PB}(\beta) := -\frac{1}{\beta} \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_{PB}(\beta) = -\frac{1}{\beta} \ln \left(2 \cosh \beta J\right) = F(\beta)$$

Note that $\lambda(\beta) = \tanh \beta J$ and therefore $|\lambda(\beta)| < 1$ for all $\beta < \infty$.

Supplement:

Normalized measure on Z_2 and group average:

$$\langle \cdot \rangle_{Z_2} := \frac{1}{2} \sum_{\sigma \in Z_2} (\cdot) \quad \text{Haar measure}$$

Harmonic analysis on Z_2 :

Z_2 has 2 UIR:

$$D^0(\sigma) = 1 \quad \text{trivial representation}$$

$$D^1(\sigma) = \sigma \quad \text{faithful representation}$$

Let

$$f_0 := \frac{1}{2} \sum_{\sigma \in Z_2} f(\sigma) D^0(\sigma), \quad f_1 := \frac{1}{2} \sum_{\sigma \in Z_2} f(\sigma) D^1(\sigma)$$

be the Fourier coefficients of the harmonic analysis for function $f(\sigma)$. Then

$$f(\sigma) = \sum_{i=0}^1 f_i D^i(\sigma)$$

Our example $f(\sigma) = e^{\alpha\sigma}$ results in

$$f_0 = \cosh \alpha, \quad f_1 = \sinh \alpha$$

Hence

$$f(\sigma) = e^{\alpha\sigma} = \cosh \alpha D^0(\sigma) + \sinh \alpha D^1(\sigma) = \cosh \alpha + \sigma \sinh \alpha$$

Our λ is in essence the Fourier coefficient of the Boltzmann factor of the next-neighbour interaction for the non-trivial UIR divided by that for the trivial UIR.