

**Problem 11:** Generators of  $SO(3)$  in  $L^2(\mathbb{R}^3)$ .

Let  $L_1, L_2$  and  $L_3$  be the generators of rotations in  $\mathbb{R}^3$  as defined in Problem 10 generating finite rotations

$$g(\vec{\alpha}) := \exp \left\{ -i \sum_{a=1}^3 \alpha^a L_a \right\} \quad \vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$$

a) Show that for small angles  $\delta\alpha^a$  the rotation matrix explicitly reads

$$g(\delta\vec{\alpha}) = 1 + \begin{pmatrix} 0 & -\delta\alpha^3 & \delta\alpha^2 \\ \delta\alpha^3 & 0 & -\delta\alpha^1 \\ -\delta\alpha^2 & \delta\alpha^1 & 0 \end{pmatrix} + O(\delta\vec{\alpha}^2)$$

b) Consider now the infinitesimal rotation of an arbitrary point  $\vec{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ .

That is  $\delta\vec{x} := \vec{x}' - \vec{x}$  with  $\vec{x}' := g(\delta\vec{\alpha})\vec{x}$ . Calculate the generators defined by

$$X_a := - \sum_{k=1}^3 U_{ak}(\vec{x}) \frac{\partial}{\partial x_k} \quad \text{with} \quad U_{ak}(\vec{x}) := \frac{\delta x_k}{\delta \alpha^a}.$$

Show that these generators are related to the angular momentum operator  $\vec{L} := \vec{x} \times \vec{p}$  acting on  $L^2(\mathbb{R}^3)$ .

c) Calculate the structural constants for the algebra formed by  $X_1, X_2, X_3$  and the associated Cartan metric and Casimir operator.

**Problem 12:** Generators of  $SO(4)$  in  $L^2(\mathbb{R}^4)$ .

Consider the following operators acting on  $L^2(\mathbb{R}^4)$  with  $(x, y, z, t) \in \mathbb{R}^4$ :

$$\begin{aligned} M_1 &:= z\partial_y - y\partial_z, & M_2 &:= x\partial_z - z\partial_x, & M_3 &:= y\partial_x - x\partial_y, \\ N_1 &:= x\partial_t - t\partial_x, & N_2 &:= y\partial_t - t\partial_y, & N_3 &:= z\partial_t - t\partial_z. \end{aligned}$$

a) Show that these operators obey the  $so(4)$  algebra

$$[M_i, M_j] = \varepsilon_{ijk} M_k, \quad [M_i, N_i] = 0, \quad [M_i, N_j] = \varepsilon_{ijk} N_k, \quad [N_i, N_j] = \varepsilon_{ijk} M_k.$$

b) Consider a new basis of this algebra defined by

$$J_i := \frac{M_i + N_i}{2}, \quad K_i := \frac{M_i - N_i}{2}$$

and show that the operators  $(J_1, J_2, J_3)$  and  $(K_1, K_2, K_3)$  separately close a  $so(3)$  algebra, that is  $so(4) = so(3) \oplus so(3)$ .