

Problem 9: Characters of $SO(3)$ and $SU(2)$.

a) From Exercise 10 we know that an arbitrary element of $SO(3)$ and $SU(2)$ can be expressed in terms of

$$R(\vec{\omega}) = e^{-i\varphi J_3} e^{-i\theta J_2} e^{-i\omega J_3} e^{i\theta J_2} e^{i\varphi J_3}, \quad \vec{\omega} = (\omega \sin \theta \cos \varphi, \omega \sin \theta \sin \varphi, \omega \cos \theta)^T.$$

Show that for any representation its character only depends on the rotation angle ω .

b) Show that for the UIR D^j in $\mathbb{C}^{2j+1}$, i.e. $d_j = 2j + 1$, the character explicitly reads

$$\chi^j(\omega) = \frac{\sin(j + \frac{1}{2})\omega}{\sin \frac{\omega}{2}}.$$

Hint: $\mathbb{C}^{2j+1} = \text{span}\{|jm\rangle\}$, where $J_3 |jm\rangle = m |jm\rangle$, $m = -j, -j+1, \dots, j-1, j$.

c) Show that

$$\chi^{j_1}(\omega) \chi^{j_2}(\omega) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \chi^j(\omega).$$

Problem 10: Generators of $SO(3)$

Let L_k ($k \in \{1, 2, 3\}$) be 3×3 matrices whose elements are defined by $(L_k)_{ij} := -i\varepsilon_{ijk}$, where ε_{ijk} is the total anti-symmetric tensor in 3 dimension. That is,

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

a) Show that these matrices fulfill the relation $[L_k, L_m] = i\varepsilon_{kmn} L_n$.

b) Show that $(L_k^2)_{mn} = \delta_{mn} - \delta_{km}\delta_{kn}$ and therefore $L_k^{2r} = L_k^2$ for $r = 1, 2, 3, \dots$

c) Show that $\exp\{-i\varphi L_k\} = 1 + L_k^2(\cos \varphi - 1) - iL_k \sin \varphi$

and calculate the explicit 3×3 matrices. Proof $\text{Tr } e^{-i\varphi L_k} = \chi^1(\varphi)$.