

Problem 6: Classes of the dihedral group D_n .

a) Let s and d be the generators of D_n obeying $d^n = e = s^2$ and $sd = d^{-1}s = d^{n-1}s$.

Show that D_n has $\frac{n}{2} + 3$ classes for even n and $\frac{n-1}{2} + 2$ classes for odd n .

Hint: Use relation $sd^k s = d^{-k} = d^{n-k}$ and construct classes with pure rotations first.

b) Construct the Character table for D_4 .

Problem 7: Orthonormal projection operators for finite groups.

Let $\{j\}$ be the complete set of all UIR of a finite group G with dimension d_j , $n = \text{ord } G$ and $\chi^j(g)$ be the character associated with the UIR j for group element $g \in G$. Show that for an arbitrary in general reducible but unitary representation D of G the operator

$$\mathbb{E}^j := \frac{d_j}{n} \sum_{g \in G} \chi^{j*}(g) D(g)$$

projects onto the unitary irreducible subspaces Σ^j of D associated with the UIR D^j .

Proof the properties $\mathbb{E}^{j\dagger} = \mathbb{E}^j$, $\mathbb{E}^j \mathbb{E}^k = \mathbb{E}^j \delta_{jk}$ and $\sum_{\{j\}} \mathbb{E}^j = 1$.

Problem 8: The dynamical $SO(4)$ symmetry of the Kepler-Problem

Consider the well-known 2-body Kepler problem describing the motion of a planet with (reduced) mass m around a central star of (total) mass M . Let $E < 0$ be the total energy of the effective bounded one-body motion characterized by

$$\frac{m}{2} \dot{\vec{r}}^2 - G \frac{Mm}{r} = E, \quad r = |\vec{r}|, \quad G \text{ is the gravitational constant.}$$

Consider now a new "time" s along the Kepler orbit defined by $ds = dt/r$, that is, $dt/ds = r =: t'$. The prime denotes the derivative with respect to the new time s . The time t will now be considered as a fourth independent degree of freedom coupled to the others via its equation of motion $t' = r$.

a) Show that the above equation for the Kepler orbit reads in the new time

$$(\vec{r}')^2 - 2GMt' = \frac{2E}{m} (t')^2.$$

b) Define the normalised 4-velocity $u := (u_0, \vec{u}) \in \mathbb{R}^4$ where $u_0 := t'/r_E$ and $\vec{u} := \vec{r}'/v_E r_E$ with $v_E^2 := 2m/|E|$ and $r_E := GM/v_E^2$. Show that the orbit of the 4-velocity is described by unit circles in $\mathbb{R}^4$ with center $(1, \vec{0})$.

c) For $E = 0$ show that the components of the 4-velocity $u_0 := t'/r_S$ and $\vec{u} := \vec{r}'/r_S c$, where $r_S := 2GM/c^2$, obey the relation $\vec{u}^2 = u_0$.

Click on above picture to run the animation!