# Measuring Visibilities with MIDI

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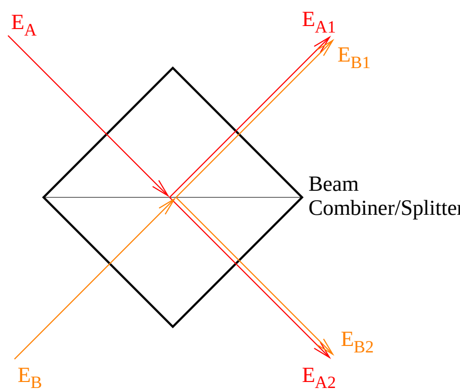
## 1 Monochromatic Point-Source

**HIGH-SENS:** Let $I_A = |\vec{E}_A|^2$ be the intensity from telescope A, $I_B = |\vec{E}_B|^2$ that from telescope B. Both go into the beam-combiner/splitter, $I_{A1}$, $I_{A2}$, $I_{B1}$, and $I_{B2}$ come out. $I_{A1}$ interferes with $I_{B1}$ and $I_{A2}$ with $I_{B2}$.




One important aspect of the beamsplitter is that it adds a phase shift of $\pi/2$ between transmitted and reflected wave (see W. Traub in "Principles of Long Baseline Stellar Interferometry", p. 48). Thus the combined field has to be written as

$$\begin{aligned} \vec{E}_1 &= \vec{E}_{A1} & &+ \vec{E}_{B1} \cdot e^{i\pi/2} \\ \vec{E}_2 &= \vec{E}_{A2} \cdot e^{i\pi/2} & &+ \vec{E}_{B2} \end{aligned}$$

The electromagnetic fields can be written as $\vec{E}_{A1} = E_{A1} e^{i\phi_A}$, $\vec{E}_{A2} = E_{A2} e^{i\phi_A}$ etc. (Note that the phase is the same).
Now let's see what we get on the detector:

$$\begin{aligned} I_1 &= |\vec{E}_1|^2 \\ &= (\vec{E}_{A1} + \vec{E}_{B1} e^{i\pi/2}) \times (\vec{E}^*_{A1} + \vec{E}^*_{B1} e^{-i\pi/2}) \\ &= |\vec{E}_{A1}|^2 + \vec{E}_{A1} \vec{E}^*_{B1} e^{-i\pi/2} + \vec{E}^*_{A1} \vec{E}_{B1} e^{i\pi/2} + |\vec{E}_{B1}|^2 \\ &= I_{A1} + E_{A1} E_{B1} e^{i(\phi_A - \phi_B - \pi/2)} + E_{A1} E_{B1} e^{-i(\phi_A - \phi_B - \pi/2)} + I_{B1} \\ &= I_{A1} + E_{A1} E_{B1} \cdot 2\cos(\phi_A - \phi_B - \pi/2) + I_{B1} \\ &= I_{A1} + 2 E_{A1} E_{B1} \sin(\Delta\phi) + I_{B1} \end{aligned}$$

$$
\begin{aligned}
I_2 &= I_{A2} - 2E_{A2}E_{B2}\sin(\Delta\phi) + I_{B2} \\
I_1 - I_2 &= I_{A2} - I_{A1} + 2\,(E_{A1}E_{B1} + E_{A2}E_{B2}) \cdot \sin(\Delta\phi) + I_{B2} - I_{B1}
\end{aligned}
$$

To first order, the intensity-terms on the right hand side are constant, therefore filtering with a high-pass filter will remove them:

$$
\begin{aligned}
I_1 - I_2 &= 2\,(E_{A1}E_{B1} + E_{A2}E_{B2}) \cdot \sin(\Delta\phi) \\
&= 2\,(\sqrt{I_{A1}I_{B1}} + \sqrt{I_{A2}I_{B2}}) \cdot \sin(\Delta\phi)
\end{aligned}
$$

If $I_{A1} = I_{A2} = I_A/2$ and $I_{B1} = I_{B2} = I_B/2$, the equation can be simplified further, but this case is too unrealistic to be considered here. Instead, we introduce the fractions of light seen through telescope A/B and channel 1/2: $f_{A1} = I_{A1}/I$, $f_{A2} = I_{A2}/I$, etc. Then

$$
\begin{aligned}
I_1 - I_2 &= 2\,(\sqrt{f_{A1}f_{B1}} + \sqrt{f_{A2}f_{B2}})I \cdot \sin(\Delta\phi) \\
&= 2fI \cdot \sin(\Delta\phi)
\end{aligned}
$$

with the photometric correction factor $f = \sqrt{f_{A1}f_{B1}} + \sqrt{f_{A2}f_{B2}}$.

Interference occurs at each pixel of the detector, therefore all the $I$'s and $\vec{E}$'s and $f$'s are functions of $x$ and $y$. Furthermore, the wavelength $\lambda$ is a function of $x$.

# 2 Monochromatic Extended Source

From the last section, we know that the fringe signal of a point source is

$$(I_1 - I_2)^{\mathrm{Point}} = 2fI \cdot \sin(\Delta\phi)$$

Where $\Delta\phi = 2\pi D_\lambda + 2\pi\theta B_\lambda$ is the phase difference introduced by baseline and delay lines. $D_\lambda = D/\lambda$ is the optical path difference introduced by the delay lines, and $B_\lambda = B/\lambda$ is the Baseline projected onto the sky, both in units of $\lambda$.
For an extended source, we have to integrate this over an angle on the sky:

$$
\begin{aligned}
(I_1 - I_2)^{\mathrm{ext}} &= \int \mathrm{d}\theta\,(I_1 - I_2)^{\mathrm{Point}}(\theta) \\
&= \int \mathrm{d}\theta\, 2fI(\theta) \cdot \sin(2\pi D_\lambda + 2\pi\theta B_\lambda) \\
&= \int \mathrm{d}\theta\, fI(\theta) \cdot \tfrac{1}{i}\,(\exp(2\pi i D_\lambda + 2\pi i\theta B_\lambda) - \exp(-2\pi i D_\lambda - 2\pi i\theta B_\lambda)) \\
&= \tfrac{1}{i} f \exp(2\pi i D_\lambda) \int \mathrm{d}\theta\, I(\theta)\exp(2\pi i\theta B_\lambda) \\
&\quad - \tfrac{1}{i} f \exp(-2\pi i D_\lambda) \int \mathrm{d}\theta\, I(\theta)\exp(-2\pi i\theta B_\lambda)
\end{aligned}
$$

The integrals can be seen as Fourier transformations:

$$\begin{aligned}\int \mathrm{d}\theta\, I(\theta) \cdot \exp(2\pi i\theta B_\lambda) = &\quad \tilde{I}(B_\lambda) &= |\tilde{I}(B_\lambda)| \cdot e^{i\phi(B_\lambda)} \\ \int \mathrm{d}\theta\, I(\theta) \cdot \exp(-2\pi i\theta B_\lambda) = &\quad \tilde{I}(-B_\lambda) &= |\tilde{I}(B_\lambda)| \cdot e^{-i\phi(B_\lambda)}\end{aligned}$$

$$\begin{aligned}(I_1 - I_2)^{\mathrm{ext}} &= \tfrac{1}{i} f e^{2\pi i D_\lambda} \cdot |\tilde{I}(B_\lambda)| \cdot e^{i\phi(B_\lambda)} - \tfrac{1}{i} f e^{-2\pi i D_\lambda} \cdot |\tilde{I}(B_\lambda)| \cdot e^{-i\phi(B_\lambda)} \\ &= f|\tilde{I}(B_\lambda)| \cdot \tfrac{1}{i} \left(\exp(2\pi i D_\lambda + i\phi(B_\lambda)) - \exp(-2\pi i D_\lambda - i\phi(B_\lambda))\right) \\ &= 2\, f|\tilde{I}(B_\lambda)| \cdot \sin(2\pi D_\lambda + \phi(B_\lambda))\end{aligned}$$

# 3 Finite Bandwidth

In reality, we cannot record a monochromatic signal, even if we consider the measurement of a single pixel. We have to integrate over the area of the pixel, which corresponds to an integral over wavelength. We write it as an integral over wavenumber $k = 2\pi/\lambda$ since this is easier to calculate:

$$\int_{k_0-\Delta k/2}^{k_0+\Delta k/2} dk\, (I_1 - I_2) = 2f|\tilde{I}(B_\lambda)| \int dk\, \sin(kD + \phi(B_\lambda))$$

Here we assumed that $f\tilde{I}(B_\lambda) \approx \mathrm{const.}$
The integral is

$$\begin{aligned}&\int dk\, \sin(kD + \phi(B_\lambda)) \\ &= \left[-\frac{\cos(kD + \phi(B_\lambda))}{D}\right]_{k_0-\Delta k/2}^{k_0+\Delta k/2} \\ &= \frac{1}{D}\left(\cos\left((k_0 - \tfrac{\Delta k}{2})D + \phi(B_\lambda)\right) - \cos\left((k_0 + \tfrac{\Delta k}{2})D + \phi(B_\lambda)\right)\right) \\ &= -\frac{2}{D} \cdot \sin(k_0 D + \phi) \cdot \sin(-\Delta k\, D/2) \\ &= \sin(k_0 D + \phi) \cdot \Delta k \cdot \frac{\sin(\Delta k\, D/2)}{\Delta k\, D/2}\end{aligned}$$

Here we used the formula

$$\cos(a) - \cos(b) = -2\sin(\tfrac{a+b}{2})\, \sin(\tfrac{a-b}{2})$$

$\sin(x)/x$ is the sinc-function, which is $1$ for $x = 0$, i.e. $D = 0$ in our case. Therefore, the amplitude of the fringe signal measured by MIDI at zero OPD is

$$2f|\tilde{I}(B_\lambda)|\Delta k \sin(\phi)$$

However, this does not help us very much, since we do not know at which position of the delay lines $D = 0$ actually is.

# 4 Power Spectrum Analysis

The signal measured by MIDI is

$$(I_1 - I_2) = 2f|\tilde{I}(B_\lambda)| \sin(k_0 D + \phi) \cdot \Delta k \cdot \mathrm{sinc}(\Delta k\, D/2)$$

## 4.1 Narrow Wavelenght Band

If the length of an OPD-scan is small compared to the coherence length, i.e. the period of the sinc $2\pi/\Delta k$, then we can neglect the sinc-term. This is the case for all reasonable wavelength-bin sizes and all scan lengths commonly used. The Fourier transform of the fringe signal consists then of two $\delta$-peaks at frequencies $\pm k_0$:

$$\mathrm{FT}\left(I_1 - I_2\right) = 2f|\tilde{I}(B_\lambda)| \cdot \Delta k \cdot \frac{1}{2i}\left(\delta(k+k_0)e^{i\phi} - \delta(k-k_0)e^{-i\phi}\right)$$

The Power spectrum is the square of the Fourier transform:

$$\mathrm{Power} = (f|\tilde{I}(B_\lambda)|\Delta k)^2 \cdot (\delta(k+k_0) + \delta(k-k_0))$$

We call the amplitude $\delta$-peaks the (square of the) correlated flux $I_{\mathrm{corr}}$:

$$I_{\mathrm{corr}} = \Delta k \cdot f|\tilde{I}(B_\lambda)|$$

(Note that this is the amplitude of the fringe signal if the sinc-term is unity.)

It is useful to define the photometric flux $I_{\mathrm{phot}} = \Delta k\, f I$, where $I$ is the total flux of the target, i.e. $\tilde{I}(0)$. MIDI does not measure the pure photometric flux, but four different approximations of it (namely $f_{\mathrm{A1}}I_{\mathrm{phot}}$, $f_{\mathrm{A2}}I_{\mathrm{phot}}$, $f_{\mathrm{B1}}I_{\mathrm{phot}}$, and $f_{\mathrm{B2}}I_{\mathrm{phot}}$). When we wish to compute the visibility, we should combine these four measurements in the same way as the instrument does it, i.e. use $fI_{\mathrm{phot}}$. The $\Delta k$ takes into account that we bin the photometric measurement, i.e. add the flux in a number of pixels.
The visibility $V$ is the normalized Fourier-transform of the object's light distribution $I(\theta)$:

$$V := \frac{\tilde{I}(u)}{\tilde{I}(0)} = \frac{I_{\mathrm{corr}}}{I_{\mathrm{phot}}}$$

## 4.2 Wide Wavelength Band

For completeness, we analyze also the case of wide wavelength bands. This applies, for example, if the full N-band is used with spectral dispersion. To calculate the Fourier transform, we make use of the convolution theorem, i.e. we Fourier transform the factors separately and convolve them afterwards. The term $f|\tilde{I}(B_\lambda)|$ is easy, since we assumed it to be constant in the last section (repeating the calculation with a non-constant $\tilde{I}$ is left as an exercise for the reader).
The sin-term yields again two delta-peaks at frequencies $\pm k_0$:

$$\mathrm{FT}\left(\sin(k_0 D + \phi)\right) = \frac{1}{2i}\left(\delta(k+k_0)e^{i\phi} - \delta(k-k_0)e^{-i\phi}\right)$$

Finally, the Fourier transform of the sinc-term is a simple top-hat function:

$$\mathrm{FT}(\Delta k \cdot \mathrm{sinc}\,(\Delta k\, D/2)) = \begin{cases} 1, & \text{if } -\frac{\Delta k}{2} < k < \frac{\Delta k}{2} \\ 0, & \text{otherwise.} \end{cases}$$

The convolution of the two delta-peaks with the top-hat leads to two top-hat-functions around $\pm k_0$:

$$\mathrm{FT}(I_1 - I_2) = \begin{cases} \frac{2f|\tilde{I}(B_\lambda)|}{2i} \cdot e^{i\phi} & \text{if } -k_0 - \frac{\Delta k}{2} < k < -k_0 + \frac{\Delta k}{2} \\ -\frac{2f|\tilde{I}(B_\lambda)|}{2i} \cdot e^{-i\phi} & \text{if } k_0 - \frac{\Delta k}{2} < k < k_0 + \frac{\Delta k}{2} \\ 0 & \text{otherwise} \end{cases}$$

The Power-spectrum consist of two peaks, too:

$$\mathrm{Power}(I_1 - I_2) = \begin{cases} (f|\tilde{I}(B_\lambda)|)^2 & \text{if } -k_0 - \frac{\Delta k}{2} < k < -k_0 + \frac{\Delta k}{2} \\ (f|\tilde{I}(B_\lambda)|)^2 & \text{if } k_0 - \frac{\Delta k}{2} < k < k_0 + \frac{\Delta k}{2} \\ 0 & \text{otherwise} \end{cases}$$

In MIA, we add up the power in one of the peaks, so we obtain

$$\int_{k_0 - \frac{\Delta k}{2}}^{k_0 + \frac{\Delta k}{2}} \mathrm{Power} = \int_{k_0 - \frac{\Delta k}{2}}^{k_0 + \frac{\Delta k}{2}} (f|\tilde{I}(B_\lambda)|)^2 = \Delta k \cdot (f|\tilde{I}(B_\lambda)|)^2$$

If we define this to be the square of the correlated flux $I_{\mathrm{corr}}$, like in the narrow-band case, then we have

$$I_{\mathrm{corr}} = \sqrt{\Delta k} \cdot f|\tilde{I}(B_\lambda)|$$

Here $I_{\mathrm{corr}}$ is not proportional to $\Delta k$, but $\sqrt{\Delta} k$. This has to be taken into account if one wishes to compute a visibility for wide wavelength bands.

## 5 Coherent Analysis

Now lets have a look what happens within the coherent analysis. Of course, the signal measured by MIDI is still the same. However, we can savely drop the sinc-term since EWS works on single-pixel columns, i.e. very narrow bandwidth and large coherence length:

$$(I_1 - I_2) = 2f|\tilde{I}(B_\lambda)|\Delta k \cdot \sin(kD + \phi)$$

(We dropped the index 0 and simply write $k$ for the central frequency of a pixel). The processing steps are:

**oir1dCompressData** integrates the detector signal in the direction perpendicular to the spectral dispersion. This improves only the signal-to-noise ratio, since the signal is constant in this direction.

**oirFormFringes** subtracts the two channels and applies a high-pass filter. The result is $(I_1 - I_2)$ given by the formula above. The high-pass filter should remove only residuals of the background and leave the fringe signal unchanged.

**oirRotataInsOpd** multiples each data point by $\exp(-ikD_i)$, where $D_i$ is the known instrumental OPD. After that, the signal can be written as

$$\begin{aligned}(I_1 - I_2)_{\mathrm{rot}} &= 2f|\tilde{I}(B_\lambda)|\Delta k \cdot \tfrac{1}{2i}\left(e^{i(kD+\phi)} - e^{-i(kD+\phi)}\right) \cdot e^{-ikD_i} \\ &= f|\tilde{I}(B_\lambda)|\Delta k \cdot \tfrac{1}{i}\left(e^{i(kD_a+\phi)} - e^{-i(k(D_a+2D_i)+\phi)}\right)\end{aligned}$$

Here $D = D_a + D_i$ with the atmospheric delay $D_a$.

**oirGroupDelay** Fourier-transforms each spectrum. However, this is a different Fourier-transform than in the power-spectrum analysis, here each individual spectrum is transformed from the frequency domain to the delay domain, i.e. the signal as a function of $k$ is transformed to a function of, say, $x$. In the power-spectrum analysis, a scan (several spectra) is transformed from the delay to the frequency domain, that is, from $D$ to $k$. The Fourier-transform here is written as

$$\begin{aligned}&\int \mathrm{d}k (I_1 - I_2)_{\mathrm{rot}} \cdot e^{-ikx} \\ &= f|\tilde{I}(B_\lambda)|\tfrac{\Delta k}{i} \int \mathrm{d}k \left(e^{i(kD_a+\phi)} - e^{-i(k(D_a+2D_i)+\phi)}\right) e^{-ikx} \\ &= f|\tilde{I}(B_\lambda)|\tfrac{\Delta k}{i} \int \mathrm{d}k \left(e^{ik(D_a-x)} e^{i\phi} - e^{-ik(D_a+2D_I+x)} e^{-i\phi}\right) \\ &= f|\tilde{I}(B_\lambda)|\tfrac{\Delta k}{i} \left(\delta(D_a - x)\, e^{i\phi} - \delta(D_a + 2D_I + x)\, e^{-i\phi}\right)\end{aligned}$$

The first $\delta$-function gives a peak at the atmospheric delay $D_a$, while the second has its peak at $-(D_a + 2D_i)$. Since $D_i$ is modulated by MIDI's piezos, the second peak is suppressed if we average a few consecutive frames, and we're left with a (hopefully) strong peak at $D_a$.

**oirRotateGroupDelay** takes the output of oirFormFringes (before the multiplication with $\exp(-ikD_i)$) and multiplies it with $D = D_a + D_i$ ($D_a$ is the result from oirGroupDelay):

$$\begin{aligned}(I_1 - I_2)_{\mathrm{rot}} &= 2f|\tilde{I}(B_\lambda)|\Delta k \cdot \tfrac{1}{2i}\left(e^{i(kD+\phi)} - e^{-i(kD+\phi)}\right) \cdot e^{-ikD} \\ &= f|\tilde{I}(B_\lambda)|\tfrac{\Delta k}{i} \cdot \left(e^{i\phi} - e^{-i(k2D+\phi)}\right)\end{aligned}$$

Again, the second $e$-term is varying rapidly due to the OPD-modulation, so if we average several frames (preferably all good frames), we have

$$(I_1 - I_2)_{\mathrm{avg}} = f|\tilde{I}(B_\lambda)|\tfrac{\Delta k}{i} \cdot e^{i\phi}$$

The modulus of this is the same correlated flux as in the power-spectrum analysis:

$$|(I_1 - I_2)_{\mathrm{avg}}| = f|\tilde{I}(B_\lambda)|\Delta k = I_{\mathrm{corr}}$$

Additionally, we can obtain the phase $\phi$, although I'm not sure yet how this works. . .