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Group Theory: Mathematical Expression of Symmetry in Physics

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Abstract: The present article reviews the multiple applications of group theory to the symmetry problems in physics. In classical physics, this concerns primarily relativity: Euclidean, Galilean, and Einsteinian (special). Going over to quantum mechanics, we first note that the basic principles imply that the state space of a quantum system has an intrinsic structure of pre-Hilbert space that one completes into a genuine Hilbert space. In this framework, the description of the invariance under a group G is based on a unitary representation of G. Next, we survey the various domains of application: atomic and molecular physics, quantum optics, signal and image processing, wavelets, internal symmetries, and approximate symmetries. Next, we discuss the extension to gauge theories, in particular, to the Standard Model of fundamental interactions. We conclude with some remarks about recent developments, including the application to braid groups.

Keywords: group theory; Lie group; symmetry; representations; quantum physics; elementary particles; braid groups

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1. Prologue

Group theory is nowadays the backbone of elementary particle physics and many other domains of physics as well. The obvious connection is, of course, the description of symmetries. However, this situation is the consequence of a long chain of evolutionary steps, which goes back, in fact, to the highest antiquity, although groups themselves were born only in the 19th century. The present paper aims to describe this long history in a pedagogical, and mathematically non-technical way. Of course, for specialists in the field of methods of group and symmetry analyses, the information presented in this review is generally known and will appear as superficial. However, for a non-specialist reader who is interested in the topic of symmetry and its applications, this review may be of some interest. Hence, this is the audience to which the paper is addressed primarily. A preliminary version may be found in [1].

Ever since the highest antiquity, symmetrical figures were considered more harmonious and more perfect. Examples can be found in all times and all cultures. To mention a few, there are the Egyptian pyramids (3000 BC), several items of Minoan art (18th century BC), jewels from the Mycenaean civilization (1000–600 BC), Platonic solids (tetrahedron, cube (hexahedron), octahedron, dodecahedron, icosahedron), and ubiquitous figures in Islamic art. A significant example is given by a Mycenaean ornament, shown in Figure 1 [2]. The artist has clearly identified the symmetry of the object, and exploits it for aesthetic reasons only.

According to Kepler, symmetry properties reflect the harmony of the world, no more. However, their systematic study required a mathematical level that was available only at the end of the 19th century), namely *group theory*, essentially invented by E. Galois. The theory rapidly enjoyed remarkable developments, thanks to such authors as G. Frobenius, I. Schur, W. Burnside, E. Cartan and H. Weyl. A survey of the theory can be found in the textbooks of Loebl [3] or Gilmore [4]. For a survey of the applications to physics, we refer the reader to the review [5].

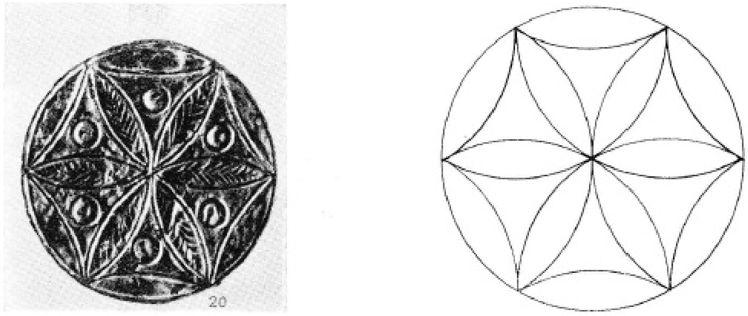

Figure 1. An antique example: symmetry of order 6 in a Mycenaean ornament. (**left**) The ornament. (**right**) Its symmetries (from [2]).

As for the organization of the paper, after a quick reminder of the relevant notions, we begin with the symmetries in classical physics, focusing on the various theories of relativity and their mutual relationship. Next, we turn to symmetries in quantum physics. After reviewing the basic principles, we examine, successively, atomic and molecular physics, quantum optics, signal processing and wavelets. Then, we turn to internal symmetries, culminating in the standard model of fundamental interactions and gauge theories. Finally we examine some recent developments, touching upon Kac–Moody algebras and braid groups, with a possible application in quantum computing. More than 40 references are given.

Group theory first entered physics with *crystallography* at the end of the 19th century. Given an arbitrary crystal, it is an easy exercise to figure out all symmetry operations that leave it unaffected: reflection through certain planes, inversion with respect to the center, rotations around given axes through the center (only the angles $2\pi/n$, with n = 2, 3, 4 or 6, will be compatible with the periodicity of the crystalline lattice), or any combination of these. A systematic investigation shows there exist exactly 32 different combinations of symmetry properties; accordingly, crystals are subdivided into 32 *crystal classes*. For a given class, the symmetry operations form a group with a finite number of elements, called a *point group*. Combining these with the lattice translations for each of the 14 different types of (Bravais) lattices, one then obtains the 230 *space groups*, a remarkable achievement, due to Fedorov (1885) and Schönflies (1891). This result is, of course, purely classical, but it illustrates the primordial rôle of group theory in physics, namely, to organize data in a rational fashion.

2. A Gentle Reminder

It may be useful to remind the reader of the basic definitions of the objects we will meet in the following, without any pretension of mathematical rigor. A group is a set G equipped with an internal composition law $(g, g') \mapsto gg'$, called product, such that the following holds:

(i) The product is associative: $g_1(g_2g_3) = (g_1g_2)g_3, \forall\, g_1, g_2, g_3 \in G$;
(ii) There exists a *neutral element* $e \in G$ such that $eg = ge = g, \forall g \in G$, necessarily unique.
(iii) Every element $g \in G$ has an *inverse* $g^{-1} \in G$ (necessarily unique) such that $gg^{-1} = g^{-1}g = e$.

The group is called *abelian* or *commutative* if the product is commutative:

$$g_1g_2 = g_2g_1, \forall\, g_1, g_2 \in G. \tag{1}$$

A *Lie group* is a group with a structure of analytic manifolds compatible with the group structure. This means that the elements $g \in G$ can be parameterized, $g \equiv g(\varphi_k)$, in such a way that the group operations are given by the analytic functions of the parameters φ_k.

The *Lie algebra* $\mathfrak{g}$ of a Lie group G is the vector space of the vector tangent at the identity. Geometrically, these are the infinitesimal generators of the group.

Finally, a (linear) *representation* of a group G is a homomorphism T of G in the invertible linear operators of a vector space:

- $T(g_1 g_2) = T(g_1)T(g_2)$,
- $T(g^{-1}) = T(g)^{-1}$,
- $T(e) = I$.

The representation T is unitary if every $T(g)$ is a unitary operator.

The groups that we will meet in the following are (almost) all Lie groups, namely, the following:

- SO(n), the rotation group of $\mathbb{R}^n$
- SU$(n) = \{\ n \times n$ unitary matrices with determinant 1$\}$
- SO(1,3), the Lorentz group, i.e., the group of isometries of $\mathbb{R}(1,3) = \mathbb{R}^4$ with metric $(+ - - -)$
- The group of Galilei transformations.
- Inhomogeneous groups (semidirect products), $\mathbb{R}^3 \rtimes SO(3)$ (Euclidean group) or $\mathbb{R}^4 \rtimes SO(1,3)$ (the Poincaré group).

An important remark is in order here. Whereas mathematicians envisage group representations in the language of operators in a Hilbert space, physicists often resort to graphical methods, largely pioneered by D. Speiser. In the spirit of E. Cartan, the accent is put rather on Lie algebras [6] to which Weyl added a geometrical flavor, the so-called weight diagrams that characterize irreducible representations [7]. A detailed discussion of the graphical method and its origin may be found in the volume of [8]. Remarkably, the method may be extended to infinite dimensional Lie algebras, the so-called Kac–Moody algebras [9].

3. Symmetries in Classical Physics

All the symmetries mentioned in the previous section pertain to the classical domain, but, of course, groups are absent since they were invented by mathematicians in the 19th century only.

Besides crystallography, the essential application of group theory in classical physics is the theory of relativity. The *principle of relativity* says that a physical system is described by the same equations in two reference frames of space-time if the latter are *equivalent*, that is, the two frames are mapped onto each other by a space-time transformation (translation, rotation, ...). Such transformations obviously form a group, called the *group of relativity*. The standard examples are the following:

- System at rest: Euclidean group.
- Special relativity: Poincaré group, i.e., inhomogeneous Lorentz group.
- General relativity: no global group, there is only *local* invariance under the Poincaré group.

A word of explanation is required here. First we recall that Einstein arrived at his special relativity by forcing mechanics to be invariant under the Lorentz group (in fact, the Poincaré group), like electromagnetism (Maxwell equations), as follows from the fact that the speed of light c is forced to be constant in any reference frame. Indeed, a Lorentz transformation becomes a Galilean transformation when $c \to \infty$. Hence, the Lorentz group becomes the (homogeneous) Galilean group in that limit.

As it is well known, invariance under a Lie group implies conservation laws and vice versa as stated by *Noether's theorem*: if a system is invariant under the Lie group G, the corresponding conserved quantities are given by elements of the Lie algebra $\mathfrak{g}$ of G (or its enveloping algebra).

4. Principles of Quantum Physics

As stated in any textbook (e.g., Cohen-Tannoudji et al. [10]), quantum mechanics rests on three principles, namely, the following:

(1) The *superposition* principle: every linear superposition of two states of a system is a state, which implies that the state space is intrinsically a vector space.
(2) The *transition amplitude* between two states is given by a Hermitian sesquilinear form: $A(\phi_{\rm in} \to \phi_{\rm out}) = \phi_{\rm out}\phi_{\rm in}$.
In the same way, the corresponding *transition probability* is given by the squared modulus of this amplitude: $P(\phi_{\rm in} \to \phi_{\rm out}) = |\langle\phi_{\rm out}|\phi_{\rm in}\rangle|^2$.
Thus, the state space $\mathcal{H}_0$ is a *pre-Hilbert space*.
(3) The observables of the system are represented by linear Hermitian operators in $\mathcal{H}_0$,.
Since these in general do not commute, this implies the presence of *uncertainty relations*.
Thus, one has to rely on a probabilistic interpretation of the theory.

Then, in order to have sufficiently powerful results, von Neumann [11] required in addition that the state space be complete, that is, a *Hilbert space*, denoted as $\mathcal{H}$. This allows to exploit a richer mathematical arsenal: self-adjoint operators, spectral theory, unitary time evolution, etc. Quite remarkably, it is in that work that von Neumann gave the first precise definition of Hilbert space.

This is not the end of the story, however. Indeed, the von Neumann formalism is rigorous, but too cumbersome for most practicing physicists. Instead, they use the so-called *Bra–Ket formalism* of Dirac [12]. Here, all points on the spectrum of an observable are treated on the same footing, whether they belong to the discrete spectrum (eigenvalues) or to the continuous spectrum. While this is convenient, it is not strictly correct. Yet, there is a way out: the Dirac formalism may be recovered in a rigorous fashion if one introduces a *rigged Hilbert space (RHS)*:

$$\Phi \subset \mathcal{H} \subset \Phi^{\times}. \tag{2}$$

where Φ is a dense subspace of $\mathcal{H}$, generated by a set of labeled observables, and $\Phi^{\times}$ is the conjugate dual of Φ. Under some mathematical conditions on Φ, the Dirac formalism is recovered, now rigorously. As for the physical interpretation, the elements of Φ represent those states that are physically preparable, whereas $\Phi^{\times}$ contains generalized states associated with measurement operations. A full discussion of the sequence von Neumann $\to$ Dirac $\to$ RHS can be found in [13–15].

5. Symmetries in Quantum Physics

5.1. General Principles

A *symmetry* is defined as a map $\mathcal{H} \to \mathcal{H}$ that preserves all transition probabilities. Then, the whole domain is based on the following two results [3,4]:

- According to Wigner [16], a symmetry is realized by a unitary or an anti-unitary operator in $\mathcal{H}$.
- According to Bargmann, a symmetry group G is realized by a unitary representation $U(G)$ in $\mathcal{H}$ (up to phases) : $U(G) = \{U(g) : \mathcal{H} \to \mathcal{H}, g \in G\}$ with the relations given in Section 2:

$$U(g_1)U(g_2) = U(g_1 g_2), \quad U(g^{-1}) = [U(g)]^{-1}, \quad U(e) = I. \tag{3}$$

If $U(G)$ is reducible, it may be decomposed into irreducibles:
$U = \oplus_j U_j$, corresponding to $\mathcal{H} = \oplus_j \mathcal{H}_j$.

Let $U = \oplus_j U_j$ and let A be a physical quantity with a simple behavior under G, e.g., a vector or a tensor if $G = SO(3)$. Such a quantity is often given by *matrix elements* $\langle\phi|A|\psi\rangle$, with $\phi \in \mathcal{H}_j, \psi \in \mathcal{H}_k$. Then, this quantity in general does not depend on the states ϕ, ψ, but only on the subrepresentations U_j, U_k, and often vanishes (selection rule). This is the content of the Wigner–Eckart theorem [10].

In addition, as mentioned above, the main observables derive from the Lie algebra of the symmetry group, thanks to Noether's theorem:

- Invariance under Euclidean operations (translations and rotations) $\Rightarrow$ total momentum and total angular momentum;

- Invariance under time translations ⇒ Hamiltonian (total energy);
- Invariance under the Galilei group ⇒ position observables.

A fundamental concept for applications is that of *approximate symmetries*, introduced by Racah [17]. Take a Hamiltonien $H = H_0 + H_1$, where H_0 is invariant under G and H_1 is a small correction, invariant only under a subgroup G_1 of G, but transforming simply under G. Then, the computation of the matrix elements is considerably simplified. Iterating the procedure, one obtains a hierarchy of approximate symmetries, more and more broken, $H = H_0 + H_1 + H_2 + \ldots$, corresponding to a hierarchy of groups $G \supset G_1 \supset G_2 \supset \ldots$. This approach, based on exact and broken symmetries, is proven to be extremely efficient in (non-relativistic) quantum physics, beginning already in the 1920s. Instead of giving the long list of physicists and mathematicians who contributed to the subject, we refer the reader to the two volumes of Loebl [3] or Gilmore [4].

5.2. Early Results: Atoms and Molecules

This method has proven to be essential in atomic and molecular physics. Take first the hydrogen atom (Coulomb potential, no spin). Energy levels are given by Balmer's formula, $E_n \sim -1/n^2, n = 1,\ 2, \ldots$, and they are degenerate: for every n, the angular momentum takes the values $l = 0,\ 1,\ 2, \ldots, n-1$, and for each l, there are $2l+1$ states indexed by $m_l = -l, -l+1, \ldots, l$.

Then group theory explains the situation as follows. For each n, the irreducible representations $D^{(l)}(l = 0,\ 1, \ldots, n-1)$ of SO(3) constitute a single irreducible representation $D^{(n^2)}$ of SO(4). This is an accidental degeneracy, and it is already present at the classical level as shown by Pauli in 1926 and Fock in 1935. If one adds the $1/2$ spin of the electron, each state $|n, l, m\rangle$ admits 2 electrons, so the angular momentum becomes $y = l \pm \frac{1}{2}$, which corresponds to the decomposition in SU(2) : $D^{(l)} \otimes D^{(1/2)} = D^{(l+1/2)} \oplus D^{(l-1/2)}$. The next step is to introduce the so-called *dynamical symmetry group* [18], which in this case may be taken as SO(4,1) or SO(4,2). Indeed, $D = \oplus_{n \in \mathbb{N}} D^{(n^2)}$ constitutes a single irreducible representation (of infinite dimension) for both groups. The same procedure may be applied for more complicated atoms, resulting in the shell model of atoms, thus finally leading to the Mendeleev table.

For *molecules*, one obtains in this way a classification of configurations and energy levels, possibly simplified through finite symmetries.

It is remarkable that, despite the fact that group theory allowed to obtain spectacular results, inaccessible so far, it was for a long time regarded with suspicion. In the 1930s, it was even dubbed "*Gruppenpest*"!

5.3. Crystals

Whereas crystallography was developed in the 19th century at the classical level, as explained in Section 1, it had to be combined with quantum mechanics in order to obtain a quantum theory of solids, as initiated in the classical paper of Bouckaert et al. in 1936 [19]. The problem is that symmetries in a crystal is a world apart from those of atoms or molecules. Indeed, if the interaction between electrons in a metal is neglected, the energy spectrum has a zonal structure. These *Brillouin zones* can also be treated via group theory. However, whereas the relevant representations form a discrete set in the case of atoms or molecules, as explained above, the representations of a space group form a continuous manifold, and must be characterized by continuously varying parameters. It follows that the energy is a continuous function of the reduced wave vector. Thus, one justifies the structure of the Brillouin zones. This was the starting point of the quantum theory of solids (condensed matter), which is now a huge domain in physics.

5.4. Optics and Photonics

The interaction between matter and light, that is, *quantum optics*, is another field where group theory plays a major role. In the spirit of Klein, consider first a quantum harmonic oscillator. Creation $a^\dagger$ and annihilation a operators generate the Lie algebra of the Weyl–Heisenberg group: $(a, a^\dagger, I) \sim (q, p, I)$, where q, p are the position and momentum operators, respectively. This approach is sufficient for treating a large number of Hamiltonians at most quadratics. Such Hamiltonians cover a substantial part of quantum optics, including lasers and other coherent phenomena. The states of such systems are the well-known (canonical) *coherent states*. They were introduced by Schrödinger in 1926 as those quantum states that best describe best the classical limit of quantum mechanics. Yet, they were quickly forgotten, thanks to a disparaging remark of Pauli. However, they were rediscovered around 1960 by R.J. Glauber, J.R. Klauder and E.C.G. Sudarshan [20] in the context of the quantum optics description of a coherent light beam emitted by lasers. Group theory had little to do in this setup, until mathematician Perelomov and physicist Gilmore, independently, discovered in 1972 that coherent states may be obtained by the action of a Lie group on a basis vector $\psi \in \mathcal{H} : \psi_g := U(g)\psi, g \in G$, where U is a unitary representation of the group G. This situation is realized in several well-known examples: the Weyl–Heisenberg group $G_{\rm WH}$ yields the canonical coherent states; the rotation group SO(3) yields the spin coherent states; the group SU(1,1) yields the coherent states describing a particle in an infinite potential well or the *squeezed states* of an atom, etc. In fact, the (generalized) coherent states have found applications in almost all domains of physics, not only quantum optics, but also nuclear and atomic physics, condensed matter physics, quantum electrodynamics (the infrared problem), quantization and dequantization, path integrals, etc. A systematic survey can be found in the textbook of J-P. Gazeau [21].

5.5. Signal Processing: Wavelets and Their Generalizations

A somewhat unexpected outcome of the coherent state formalism is the spectacular development of *wavelet analysis*. Indeed, continuous wavelets are nothing but coherent states generated by the affine or $ax + b$ group (dilations and translations on $\mathbb{R}$). They were discovered (in dimension 1) by A. Grossmann and J. Morlet in 1984, in the context of oil exploration by seismic techniques. The two authors were quickly joined by I. Daubechies, and wavelets were applied to the analysis of (analogic) signals.

In practice, the success of wavelets rests on the fact that they improve the capabilities of the Fourier transform, a standard tool in physics and engineering for computing the spectrum of a signal. Indeed, Fourier is a global transform that loses all information about the localization in the signal, whereas the wavelet transform (WT) is local and allows to keep information simultaneously on position and energy. In order to achieve this, one has to replace the purely frequency domain representation of the Fourier transform by a *time frequency representation*. The wavelet transform is an example, but there are other ones. Notably the Gabor transform, also called Windowed or Short Time Fourier transform (STFT). Both obey the coherent state formalism, with the affine or $ax + b$ group for the WT and the Weyl–Heisenberg group for the STFT. Thus, the wavelets corresponding to the latter, sometimes called gaborettes, are simply the canonical coherent states.

On the mathematical side, the WT has led to a remarkable development, being extended to arbitrary square integrable representations of any locally compact group [22]. A related formalism is *coorbit theory*, based on an integrable representation of a group, which yields a very general and elegant discretization of the continuous analysis of a signal [23].

In signal processing, the next step was made by Y. Meyer and S. Mallat, who introduced *discrete wavelets* by a particular type of discretization (a well-known technique in coherent state analysis). These new wavelets allowed the treatment of digital signals (no more group theory here!) and the domain exploded in the signal community. This formalism can be generalized to higher dimensions, leading to a whole menagerie of specialized transforms (ridgelets, curvelets, contourlets, etc.).

On the (applied) physics side, the WT was extended to two dimensions by A. Grossmann, I. Daubechies, and R. Murenzi et al. in 1990, with the aim of treating image processing. In that context, an important aspect to detect is the directional features present in an image (ridges, roads, etc.) for which one needs directional wavelets. Whereas these can be designed in the pure 2D WT framework [24], a more radical solution is that of *shear-*

lets, introduced by Kutyniok and her collaborators [25]. The idea is simply to replace the rotations inherent to the 2D continuous WT by *shear operations*, which generates extremely directional wavelets. The interesting fact is that, like the WT, the shearlet transform stems from a Lie group, called the *shearlet group*. The resulting transform may then be discretized, which yields a very powerful tool generalizing the 2D discrete WT.

There are, of course, plenty of books about wavelets, too many to cite here. Two essential ones are those of Daubechies [26] and Mallat [27]. In addition, we refer to the compendium [28], which collects all the relevant early papers in the domain. As for the connection with coherent states, we may quote our textbook [24].

6. Internal Symmetries

6.1. Discrete Symmetries

All the symmetries we have met so far pertain to Lie groups, thus are continuous, even analytic. Yet, there are also some discrete symmetries that play a crucial role, namely conjugations (involutions). Three of them are essential: C—charge conjugation, which exchanges particles and antiparticles; P—parity, which exchanges left and right; and T—time reversal, which amounts to rewind the film backward. Whereas C and P are unitary in any field theory, T is anti-unitary, and all three have a square equal to I . Concerning the status of these operations, it appeared quickly that C was broken in weak interactions. As for the product CP, it was thought for a long time that it would be conserved, but the decay of K mesons finally proved that that was not the case. Thus, only CPT remains, which is now considered to leave all interactions invariant, which is really a universal symmetry.

6.2. Continuous Symmetries

Besides symmetries of a geometrical nature, there are also continuous *internal symmetries*, which, over the years, have played a more and more essential role. The first example is *isospin*, introduced by Heisenberg in 1932 and realized by the group SU(2). For example, the nucleon, i.e., the pair proton–neutron, is a doublet of isospin $1/2$. This was the starting point of the classification of elementary particles, as we will see below.

Another precursor for internal symmetries is Wigner in his fundamental 1939 paper on representations of the Poincaré group [29]. Reducing these representations to those of the so-called *little groups*, he obtained an O(3)-like spin symmetry for massive particles, and an E(2)-like symmetry for massless particles with the helicity and gauge degrees of freedom. It is worth recalling here that, according to Dirac [30], relativistic dynamics is in fact a representation of the Poincaré group.

Let us now go back to the classification of elementary particles. First, particles are organized into isospin multiplets. Next, one adds a new internal degree of freedom, called hypercharge Y (or equivalently *strangeness* S), introduced by Gell-Mann and leading to the well-known relation $Q = T_3 + \frac{1}{2}Y$, where Q denotes the electric charge and T_3 the third component of isospin. Combining isospin and hypercharge, Gell-Mann introduced the group SU(3), which allowed to classify all particles known in 1962 in corresponding multiplets [31,32]. The most spectacular success of this approach was evidently the discovery in 1964, with the correct mass and quantum numbers, of the quasi-stable particle Ω, predicted by SU(3) .

A turning point in the theory was the suggestion, by Gell-Mann and Zweig, independently, that all known hadrons are bound states of three fundamental particles, called *quarks* (or aces), namely up, down and strange. While this model was deemed naive, it met with a remarkable success (although free quarks themselves were never seen, they are confined). Over the years, three additional quarks have been discovered: charm (1974), bottom (1977), and top (1993). Thus, we obtained the so-called *standard model* of today, with the symmetry group U(1)$\otimes$SU(2)$\otimes$SU(3).

In fact, group-theoretical techniques not only lead to the classification of particles, but they also determine their dynamical properties. In a first step, Gell-Mann and Feynman considered the electromagnetic current and the weak current as an isospin triplet. Next, Cabibbo suggested to treat all hadronic currents in the same way, by extending the model from SU(2) to SU(3). This led Gell-Mann to the charge algebra, corresponding to the symmetry SU(3)$\otimes$ SU(3). Finally, Gell-Mann postulated that the currents themselves possess the same symmetry, thus obtaining the celebrated *current algebra* (chiral symmetry), which enjoys a local symmetry SU(3)$\otimes$ SU(3). Many of the original articles are collected in the book by Dyson [33]; see also Bohm et al. [18].

With hindsight, clearly the way in which group-theoretical techniques have been used has been turned upside down. Contrary to the traditional applications, for instance, in atomic physics, the precise structure of the hadronic currents is unknown; only their symmetry matters. This is a nice analogy to the celebrated "Cheshire cat" of Lewis Carroll: the cat has vanished, and only its smile remains.

7. Gauge Theories

7.1. Evolution of the Theory

The most remarkable evolution during the last years is the emergence, then gradually the omnipresence, of gauge theories. To obtain the idea, we notice that an internal symmetry may be *global*, or *local*. In the global case, this means that the action of a group G on a quantum field $\phi(x)$ is independent of the point x; for a local symmetry, the action of G varies from point to point. In the latter case, we obtain the *gauge field theory*, and G is called the gauge group. The original idea extends back to Weyl, who in 1918 treated the electromagnetism as a U(1) gauge theory (hence, abelian). However, the real starting point was the proposition by Yang and Mills in 1954 of a nonabelian gauge theory based on SU(2). This proposal also marks the apparition of differential geometry in quantum physics (with notions such as fiber bundles, connections, etc.). However, the suggestion of Yang and Mills became popular only when the Dutch physicist G. 't Hooft proved in 1971 that a nonabelian gauge theory may be renormalizable, i.e., it may yield finite, verifiable predictions.

An important aspect is that a gauge theory is necessarily exact, which implies that it contains less arbitrary parameters and it is, therefore, more coherent. In particular, the interaction Lagrangian is determined uniquely. In addition, interactions are mediated by massless particles, such as the photon for electromagnetism and gluons for the strong interaction. A systematic description of gauge theories may be found in [34].

7.2. The Standard Model

The idea was quickly generalized. First, S. Weinberg, A. Salam et al. and S. Glashow reformulated in 1962–1968 the electroweak interactions as a gauge theory based on G = SU(2)$\otimes$U(1). Next, D. Gross, H. Politzer and F. Wilczek did the same in the 1970s with the strong interactions, thus obtaining quantum chromodynamics, based on the gauge group G = SU(3). The latter acts on a new internal degree of freedom, called *color*: each quark appears under three different colors. The end result is the present day *standard model*, which is a SU(3)$\otimes$SU(2)$\otimes$U(1) gauge theory. Actually, the model remained incomplete for a long time since an essential ingredient was missing, the Higgs–Brout–Englert boson. This boson is supposed to give a nonzero mass to all particles, except the photon and the gluons, via a subtle mechanism called spontaneous symmetry breaking (i.e., the ground state has a symmetry smaller than the Hamiltonian). The HBE boson was finally discovered at CERN in 2012, which resulted in the 2013 Nobel prizes for P. Higgs and F. Englert, R. Brout having passed away in the meantime.

8. Recent Developments

Besides the successful approach of the standard model, many other schemes based on group theory have been proposed, but in general, they failed. For instance, the so-called models of grand unification, based on SU(5), SO(10), etc., that aimed at describing all interactions, except gravity, never survived because they predicted new particles and/or the decay of the proton, which were never observed.

A still active proposal is *supersymmetry*, which seeks to unify bosons and fermions. Since they are based on different statistics, Bose–Einstein for bosons and Fermi–Dirac for fermions, such a theory necessarily implies the presence of anticommuting variables. This, in turn, induces a whole collection of new structures, supergroups, Lie superalgebras, supermanifolds, etc. Interesting mathematical results follow, but no physical confirmation has been obtained so far, although supersymmetry is still envisaged in quantum gravity (see below). The point is that a supersymmetric theory predicts a whole family of new particles that mirror the known ones, (photinos, gluinos,...), which have not been observed.

A different approach follows from the original idea of Weyl, who imposed the invariance under a redefinition of the length parameter, i.e., scale invariance. In two dimensions, this leads to invariance under conformal transformations. Indeed, conformal field theories have become popular (also in statistical mechanics). They are the basis of string theory (superstrings) in which the elementary constituents of matter are no longer pointlike, but rather one-dimensional objects (strings) [35]. However, this extension also requires the use of some exotic Lie groups, such as SO(32), the exceptional groups E(6), E(7), E(8) or, more generally, loop groups. On the other hand, the 2D conformal group is not a Lie group since it is infinite dimensional. In this way, infinite dimensional Lie algebras entered the physics of fundamental interactions. First came the Virasoro algebra, very close to conformal algebra. Next, by combination with the classical string theory, one can obtain the whole family of *Kac–Moody algebras* and their representations [9].

As a matter of fact, Lie algebras or their generalizations play a significant role in symmetry considerations of various physical systems. A good source of such applications is the volume of Adler et al. [36], for instance, the Lie algebra-based integrability of dynamical systems, or the symplectic and Poisson symmetries of Hamiltonian systems. Another interesting topic is the classification of singularities along Lie algebraic Dynkin diagrams, for instance, ADE singularities (here, ADE refers to all simply laced diagrams, thus of types A_n, B_n or E_n). These also appear in models of phase transitions (Landau theory).

Another instance of deformation of classical Lie algebras is that of *quantum groups*. However, one may jokingly say that quantum groups are neither groups, nor quantum, as they are, in fact, Hopf algebras, a well-known structure in algebra.

The ultimate extension of our quest is *quantum gravity*, which aims at combining quantum mechanics and general relativity. To that effect, it incorporates notions from gauge theories, supersymmetry, and superstrings. Many variants have been proposed, but no convincing candidate has emerged [37].

We conclude this survey with a totally different application of group theory, namely, the use of *braid groups*. This requires some explanation since this notion might be unfamiliar. Given a set $A_n = \{a_1, \dots, a_n\}$ of n points, a *braid* with n strands is a continuous bijection $\sigma : A_n \to A_n$. Figure 2 shows some examples with 3 or 4 strands. The composition of two braids is simply their successive application (Figure 3). With this operation, the set of all braids with n strands is a group, noted as B_n.

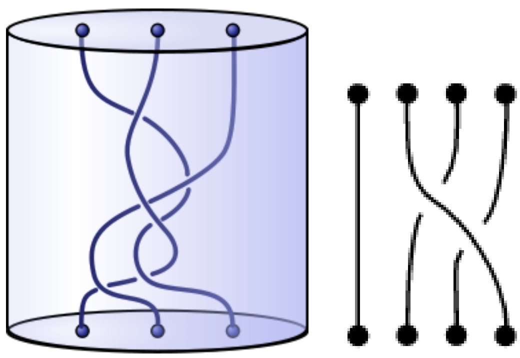

Figure 2. A braid with 3 strands, resp. 4 strands (from Wikipedia).

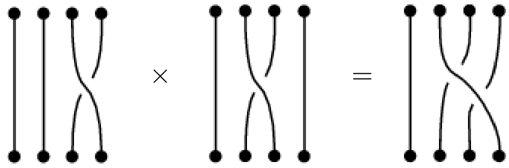

Figure 3. Product of two braids (from Wikipedia).

This notion and its representations were introduced by E. Artin in 1925 [38] in a purely mathematical context (algebraic geometry, and knot theory). More recently, they have found many applications in mathematical physics, for instance, in statistical mechanics (Yang–Baxter equation), in particle physics (anyons and Majorana fermions), in fluid mechanics, and in theoretical computer science (*quantum computing*). For these applications, we refer the reader to the review article of Kauffman [39]. Concerning *quantum computing*, another type of group appeared recently, namely, Galois groups and, more generally, profinite groups (i.e., Hausdorff topological groups, compact and totally discontinuous), which show up in Galois or finite rings theories. In particular, the p-adic group $\mathbb{Z}_p$, the typical example of a Galois group, plays an explicit role in quantum computing. For all this, we may refer to the works of Vourdas [40,41] and, for Galois theory, the book by Kibler [42].

9. Epilogue

In conclusion, it seems fair to say that group theory has grown into one of the essential tools of contemporary physics. Besides its fundamental role in relativity, it has provided physicists with a remarkable analyzing power for exploiting known symmetries, and with a considerable predictive capability, precisely in cases where the basic physical laws are unknown. Furthermore, in addition to its crucial importance in the description of fundamental interactions and elementary particles, group theory has pervaded all fields of physics, often in an unexpected way. Except for calculus and linear algebra, no mathematical technique has been so successful.

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