Exercise 15: Summary on Wiener process in $Q = \mathbb{R}^d$

①

Transition p.d.:
$$W_t^{(d)}(\vec{q}_2, \vec{q}_1) = (2\pi t)^{-d/2} \exp\left\{-\frac{(\vec{q}_2-\vec{q}_1)^2}{2t}\right\}$$

Generator:
$$T_{W^{(d)}} := \frac{1}{2}\vec{P}^2, \qquad \vec{P} = -i\vec{\nabla}$$

Feynman-Kac formula:
$$\langle \vec{q}_b | \exp\left\{-t\left(\frac{\vec{P}^2}{2} + V(\vec{Q})\right)\right\} | \vec{q}_a \rangle =$$
$$\int_{\mathcal{C}(\mathbb{R}^d, \vec{q}_a)} dW^{(d)}[\vec{x}]\; \delta(\vec{x}(t) - \vec{q}_b) \exp\left\{-\int_0^t d\tau\, V(x(\tau))\right\}$$

In particular for $V=0$
$$W_t^{(d)}(\vec{q}_b, \vec{q}_a) = \langle \vec{q}_b | e^{-t\vec{P}^2/2} | \vec{q}_a \rangle \quad \text{euclidean free propagator on } \mathbb{R}^d$$

↗ Section II.3 with $m = \hbar = 1$

Homework 14 trivially extends to $d \geq 1$

For $V(\vec{Q}) = \frac{1}{2}\omega^2 \vec{Q}^2$

$$\int_{\mathcal{C}(\mathbb{R}^d, \vec{q}_a)} dW^{(d)}[\vec{x}]\; \delta(\vec{x}(t) - \vec{q}_b) \exp\left\{-\int_0^\infty d\tau\, \frac{\omega^2}{2}\vec{x}^2(\tau)\right\} =$$
$$= \left(\frac{\omega}{2\pi \sinh \omega t}\right)^{d/2} \exp\left\{-\frac{\omega}{2}(\vec{q}_b^{\,2} + \vec{q}_a^{\,2}) \coth \omega t + \frac{\omega}{\sinh \omega t}\vec{q}_b \cdot \vec{q}_a\right\}$$

↗ Homework 7 in Euclidean time

Exercise 16: Bessel processes on $Q = \mathbb{R}^+$

Transition p.d.:
$$b_t^{(\nu)}(q_2, q_1) := \frac{q_2}{t}\left(\frac{q_2}{q_1}\right)^{\nu} \exp\left\{-\frac{1}{2t}(q_2^2+q_1^2)\right\} I_\nu\left(\frac{q_2 q_1}{t}\right)$$

I_ν: modified Besselfunction of 1. kind $\qquad q_1, q_2, \nu \geq 0$

$$I_\nu(z) := \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_{-1}^{+1} dt\, e^{\pm zt} (1-t^2)^{\nu-\frac{1}{2}} \geq 0 \quad \forall z > 0$$

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\left(1 + O(z)\right)$$

$$I_\nu(z) = \frac{1}{\sqrt{2\pi z}} e^{+z}\left(1 + O(1/z)\right) \Rightarrow \lim_{z\to\infty} \sqrt{2\pi z}\, e^{-z} I_\nu(z) = 1$$

obeys Bessel diff. eq. $\quad z^2 I_\nu''(z) + z I_\nu'(z) - (z^2+\nu^2) I_\nu(z) = 0$

Note:
$$b_t^{(\nu)}(q_2, q_1) = \sqrt{2\pi q_2 q_1/t}\; e^{-q_2 q_1/t}\, I_\nu(q_2 q_1/t) \left(\frac{q_2}{q_1}\right)^{\nu+\frac{1}{2}} w_t^{(1)}(q_2, q_1)$$

All required properties for $b_t^{(\nu)}$ follow in essence from $w_t^{(1)}$ plus the modified Weber formula for the Chapman-Kolmogorov relation

Generator:
$$T_{B^{(\nu)}} = \frac{P^2}{2} + i(\nu+\tfrac{1}{2})\, P Q^{-1} \qquad P = -i\,\partial_q$$

Δ Wiener process on half line with drift potential $V(q) = -(\nu+\frac{1}{2}) \ln q$

Fokker-Planck eq.:

$$\partial_t\, b_t^{(\nu)}(q,q_0) = \frac{1}{2}\left(\partial_q^2 - \partial_q \frac{2\nu+1}{q}\right) b_t(q,q_0)$$

in essence reduced to Bessel diff. eq.

Relation with Wiener process:

$$b_t^{(\frac{d}{2}-1)}(q_2,q_1) = q_2^{d-1} \int_{S^{d-1}} d\Omega_{\vec{q}_2}\, W_t^{(d)}(\vec{q}_2,\vec{q}_1)$$

with $d^d\vec{q}_2 = dq_2\, q_2^{d-1}\, d\Omega_{\vec{q}_2}$ ← angular part

For functionals depending only on $|\vec{x}| = x$

$$\int_{\mathcal{C}(\mathbb{R}^d,\vec{q}_1)} dW^{(d)}_{[\vec{x}]}\, F[|\vec{x}|] = \int_{\mathcal{C}(\mathbb{R}^+,q_1)} dB^{(\frac{d}{2}+1)}\, F[x]$$

Radial HO:

$$\int_{\mathcal{C}(\mathbb{R}^+,q_a)} dB_\nu[x]\, \delta(x(t)-q_b)\, \exp\left\{-\frac{\omega^2}{2}\int_0^t d\tau\, x^2(\tau)\right\} =$$

$$= q_b \left(\frac{q_b}{q_a}\right)^\nu \frac{\omega}{\sinh\omega t} \exp\left\{-\frac{\omega}{2}(q_b^2+q_a^2)\coth\omega t\right\} I_\nu\left(\frac{\omega q_a q_b}{\sinh\omega t}\right)$$

↗ Section I.10

Exercise 17: Path integration for central potentials

$$H = \frac{\vec{p}^{\,2}}{2} + V(q) \quad , \quad q = |\vec{q}|$$

without proof:

$$\int\limits_{\mathcal{C}(\mathbb{R}^d, \vec{q}_a)} dW^{(d)}[\vec{x}] \; C_\ell^{(\frac{d}{2}-1)}\left(\cos(\vec{x}(t), \vec{q}_b)\right) e^{-\int_0^t d\tau\, V(x(\tau))} =$$

$$= C_\ell^{\frac{d}{2}-1}\left(\cos(\vec{q}_b, \vec{q}_a)\right) \int\limits_{\mathcal{C}(\mathbb{R}^+, q_a)} dB^{\frac{d}{2}+\ell+1}[x] \left(\frac{q_a}{x(t)}\right)^\ell e^{-\int_0^t d\tau\, V(x(\tau))}$$

↑ Gegenbauer polynomial

follows in essence from Gegenbauer expansion formula (↗ p. 23)

⇝

$$\langle \vec{q}_b | e^{-tH} | \vec{q}_a \rangle = \int dW^d[x]\, \delta(x(t) - q_b)\, e^{-\int_0^t d\tau\, V(x(\tau))}$$

$$= \frac{\Gamma(\frac{d}{2}-1)}{2\pi^{d/2} q_b^{d-1}} \sum_{\ell=0}^{\infty} \left(\frac{d}{2}+\ell-1\right)\left(\frac{q_a}{q_b}\right)^\ell C_\ell^{\frac{d}{2}-1}\left(\cos(\vec{q}_a, \vec{q}_b)\right)$$

$$\int dB^{(\frac{d}{2}+\ell-1)}[x]\; \delta(x(t) - q_b)\, e^{-\int_0^t d\tau\, V(x(\tau))}$$

Change of Index

$$\overset{\text{via}}{=} \int dB^{(\frac{d}{2}-1)}[x]\; \delta(x(t) - q_b) \exp\left\{-\int_0^t d\tau \left(V(x(\tau)) + \frac{\ell(\ell+d-2)}{2x^2(\tau)}\right)\right\}$$

radial Path integral
with centrifugal potential

Change of Index follow directly from

Cameron – Martin – Girsanov formula

↗ Exercise 13 of SUSY lecture

Here

$$\int dB_t^{(\nu)}[x]\; F(x) = \int dB^{\mu}[x] F(x) \left(\frac{x(t)}{q_a}\right)^{\nu-\mu} e^{-\int_0^t d\tau \frac{\nu^2-\mu^2}{2x^2(\tau)}}$$

Exercise 19: Substitution formula for path integrals

Aim is to arrive at an identity of the form

$$\int_{\mathcal{C}(Q,q_a)} dM[x]\, F[x] = \int_{\mathcal{C}(R,r_a)} dN[x]\, F[TKy]$$

where

K : is a pointwise transformation of paths $\mathcal{C}(R,r_a) \to \mathcal{C}(Q,q_a)$

T : reparametrisation of paths via a "new" time, which may depend on the path

	config space	element		path space	path	path measure
$K \downarrow$	R	r	$K\downarrow$	$\mathcal{C}(R,r_a)$	$y(s)$	N
	Q	q		$\mathcal{C}(Q,q_a)$	$(Ky)(s)$	—
$T \downarrow$	Q	q	$T\downarrow$	$\mathcal{C}(Q,q_a)$	$x(t)$	M

Let

$$k: \begin{array}{l} R \to Q \\ r \mapsto q = k(r) \end{array}$$ surjective and diff. map

$\rightsquigarrow$ Induces mapping of paths

$$K: \begin{array}{l} \mathcal{C}(R, r_a) \to \mathcal{C}(Q, q_a) \\ y(s) \longmapsto (Ky)(s) := k(y(s)) \end{array}$$ pointwise path transformation

Let $$t_{Ky}: \begin{array}{l} \mathbb{R}^+ \to \mathbb{R}^+ \\ s \mapsto t = t_{Ky}(s) \end{array}$$ such that $t_{Ky}(s)$ is strictly monot. Increasing and $\lim_{s \to \infty} t_{Ky}(s) = \infty$

That is each path receives its "own" time

Is assured if we choose a τ

$$\tau: \begin{array}{l} Q \to \mathbb{R}_+ \\ Ky \mapsto \tau(Ky) \end{array}$$ being a strictly positive function

and then define $$t_{Ky}(s) := \int_0^s ds' \, \tau((Ky)(s'))$$

Note $$\frac{dt_{Ky}}{ds} = \tau((Ky)(s)) > 0$$ $\rightsquigarrow$ invertible function $s_{Ky}(t)$ exist

such that $t_{Ky}(s_{Ky}(t)) = t$

$$\Rightarrow \quad T: \begin{array}{l} \mathcal{C}(Q, q_a) \to \mathcal{C}(Q, q_a) \\ (Ky)(s) \longmapsto x(t) \equiv (TKy)(t) := (Ky)(s_{Ky}(t)) \end{array}$$

Summary : "Space-Time transformation"

$$\mathcal{C}(\mathcal{R}, r_a) \to \mathcal{C}(Q, q_a)$$

$$TU: \quad y(s) \mapsto x(t) = (TUy)(t)$$

has 2 degrees of freedom: $q := h(r)$ space transformation

$\frac{dt}{ds} = \tau((Uy)(s))$ "local" time

Transformation Lemma:

Let TU be space-time transformation $\mathcal{C}(\mathcal{R}, r_a) \to \mathcal{C}(Q, q_a)$ and N and M be two Markov processes obeying condition

$$m_t(q_2, h(r_1)) = \int_{\mathcal{C}(Q, h(r_1))} dM[x]\, \delta(x(t) - q_2) = \int_{\mathcal{C}(\mathcal{R}, r_1)} dN[y]\, \delta((TUy)(t) - q_2) \quad (*)$$

$\forall r_1, q_2$ and $t \geq 0$

then for (suitable) functionals $F: \mathcal{C}(Q, q_a) \to \mathbb{R}$ following transformation relation is valid

(**)

$$\boxed{\int_{\mathcal{C}(Q, q_a)} dM[x]\, F[x] = \int_{\mathcal{C}(\mathcal{R}, r_a)} dN[y]\, F[TUy] \quad \text{with } q_a = h(r_a)}$$

In practice the proof of (*) via its Laplace transformed is much simpler

Proof via cylinder functionals

Application to Feynman-Kac functionals

$$F[x] = \delta(x(t) - q_b) \exp\left\{-\int_0^t d\tau\, V(x(\tau))\right\}$$

Consider Green's function for $H := T_M + V(Q)$

$$\langle q_b | \frac{1}{H-E} | q_a \rangle := \int_0^\infty dt\, e^{Et} \langle q_b | e^{-tH} | q_a \rangle \qquad E < \inf \operatorname{spec}(H)$$

Euclid. propagator

$$= \int_0^\infty dt\, e^{Et} \int_{\mathcal{C}(Q,q_a)} dM[x]\, \delta(x(t) - q_b) \exp\left\{-\int_0^t d\tau\, V(x(\tau))\right\}$$

$$\overset{(*)}{=} \int_0^\infty dt\, e^{Et} \int_{\mathcal{C}(\mathbb{R},r_a)} dN[y]\, \delta\big((\kappa y)(s_{\kappa y}(t)) - q_b\big) \exp\left\{-\int_0^t d\tau\, V\big((\kappa y)(s_{\kappa y}(\tau))\big)\right\}$$

Now do pathwise substitution $\tau = s_{\kappa y}(\sigma)$ with $\frac{d\tau}{d\sigma} = \tau((\kappa y)(\sigma))$

$$= \int_{\mathcal{C}(\mathbb{R},r_a)} dN[y] \int_0^\infty ds\, \tau((\kappa y)(s))\, e^{E t_{\kappa y}(s)}\, \delta\big((\kappa y)(s) - q_b\big)$$

$$\times \exp\left\{-\int_0^s d\sigma\, \tau((\kappa y)(\sigma))\, V((\kappa y)(\sigma))\right\}$$

$$= \int_0^\infty ds\, \tau(q_b) \int_{\mathcal{C}(\mathbb{R},r_a)} dN[y]\, \delta\big((\kappa y)(\sigma) - q_b\big) \exp\underbrace{\left\{-\int_0^s d\sigma\, \tau((\kappa y)(\sigma))\,[V(\kappa y)(\sigma) - E]\right\}}_{\overset{!!}{=}\ \tilde{V}_E(y)}$$

this is in essence the classical path with
$\tau = f'(y)^2$, $(\kappa y) = f(y)$ $\rightarrow$ 31

Comment 1: Special case $V \equiv 0$ reads

$$\int_0^\infty dt\, e^{Et}\, m_t(q_b, k(r)) = \tau(q_b) \int_0^\infty ds \int_{\mathcal{C}(\mathbb{R}, r_a)} dW[y]\, \delta(k(y(s)) - q_b)$$
$$\times \exp\left\{ E \int_0^s d\sigma\, \tau(k(y(\sigma)))\right\} \qquad (*)^{LT}$$

This must be verified! Is Laplace transform of $(*)$

Comment 2:

In general k may not be injective, i.e. q may have several inverse images

$$r_q(\lambda) \quad \Rightarrow \quad k(r_q(\lambda)) = q \quad \forall \lambda \in \Lambda_q \text{ set of inverse images}$$

$$\rightsquigarrow \quad \delta(k(y(s)) - q) =: \int_{\Lambda_q} d\lambda\; J_q(\lambda)\, \delta(y(s) - r_q(\lambda))$$

generalised substitution formula of δ-function

Example: $\delta(x^2 - a^2) = \frac{1}{2|a|}\left(\delta(x-a) + \delta(x+a)\right)$ discrete set Λ_{a^2}