Exercise 13: Remarks on Witten index and SUSY QM

- Bosonic degree of freedom:

$$a := \frac{1}{\sqrt{2}}(Q + iP) \qquad \text{acting on } \mathcal{H}_B := L^2(\mathbb{R})$$

$$[a, a^+] = 1$$

$$N_B := a^+ a \qquad \text{spec } N_B = \{0, 1, 2, 3, \dots\}$$

- Fermionic degree of freedom:

$$f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{acting on } \mathcal{H}_F := \mathbb{C}^2$$

$$\{f, f^\dagger\} = 1$$

$$N_F := f^\dagger f \qquad \text{Spec } N_F = \{0, 1\}$$

- SUSY oscillator: on $\mathcal{H} := \mathcal{H}_B \otimes \mathcal{H}_F$

$$H_{susy} := \hbar\omega \left(a^+ a \otimes 1 + 1 \otimes f^\dagger f\right)$$

$$= \hbar\omega \begin{pmatrix} a^+ a & 0 \\ 0 & a^+ a \end{pmatrix} + \hbar\omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \hbar\omega \begin{pmatrix} a a^+ & 0 \\ 0 & a^+ a \end{pmatrix}$$

$$=: \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \begin{matrix} \mathcal{H}^+ & \text{spec } H_+ = \{\hbar\omega n \mid n \in \mathbb{N}\} \\ \mathcal{H}^- & \text{spec } H_- = \{\hbar\omega n \mid n \in \mathbb{N}_0\} \end{matrix}$$

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

- Witten-index:

$$\Delta := \mathrm{Tr}\left((-1)^{N_F}\, e^{-\beta H_{susy}} \right)$$

$$= \mathrm{Tr}_{\mathcal{H}^+}\left((-1)^1 e^{-\beta H_+} \right) + \mathrm{Tr}_{\mathcal{H}^-}\left((-1)^0 e^{-\beta H_-} \right)$$

$$= \mathrm{Tr}_{\mathcal{H}^-} e^{-\beta H_-} - \mathrm{Tr}_{\mathcal{H}^+} e^{-H_+} = 1$$

$$= n_- - n_+$$ $n_\pm :=$ # zero energy eigenstates of $H_\pm$

- Fredholm-index: (formally) A linear operator $\mathcal{H} \to \mathcal{H}$

$$\mathrm{ind}\, A := \dim \mathrm{Ker}\, A - \dim \mathrm{Ker}\, A^\dagger$$

$$= \dim \mathrm{Ker}\, A^\dagger A - \dim \mathrm{Ker}\, A A^\dagger$$

let $A = a$

$$\rightsquigarrow \quad \mathrm{ind}\, a = \dim \mathrm{Ker}\, H_- - \dim \mathrm{Ker}\, H_+$$

$$= n_- - n_+ = \Delta$$

Exercise 14: Pseudoclassical mechanics

Classical mechanics with Grassmann-valued degrees of freedom

Grassmann algebra:

Generators: $\psi_0, \bar{\psi}_0$ such that $\psi_0^2 = 0 = \bar{\psi}_0^2$

and $\{\bar{\psi}_0, \psi_0\} = 0$

anti commute

General element:

$$B := \underset{\substack{\downarrow \\ \text{body}}}{a_1} + \underbrace{a_2 \psi_0 + a_3 \bar{\psi}_0 + a_4 \bar{\psi}_0 \psi_0}_{\text{soul}}, \quad a_i \in \mathbb{C}$$

$$\bar{B} = a_1^* + a_2^* \bar{\psi}_0 + a_3^* \psi_0 + a_4^* \bar{\psi}_0 \psi_0$$

Bosonic degree of freedom:

$$X = X_B + q \bar{\psi}_0 \psi_0 \qquad \text{even} \quad X_B, q \in \mathbb{R}$$

Fermionic degree of freedom:

$$\psi = a \psi_0 + b \bar{\psi}_0$$

$$\bar{\psi} = a^* \bar{\psi}_0 + b^* \psi_0$$

$a, b \in \mathbb{C}$

Simplest non-trivial model

$$L(\dot{x}, x, \dot{\bar{\psi}}, \psi, \dot{\psi}, \bar{\psi}) := \frac{1}{2}\dot{x}^2 - V_1(x) + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - V_2(x)\bar{\psi}\psi$$

real valued — real valued

Homework: Equations of motion

$$\ddot{x} = -V_1'(x) - V_2'(x)\bar{\psi}\psi$$

$$\dot{\bar{\psi}} = i V_2(x)\bar{\psi} \quad , \quad \dot{\psi} = -i V_2(x)\psi$$

Ansatz: $x(t) = x_B(t) + q(t)\bar{\psi}_0\psi_0$

with $\psi(0) = \psi_0 \quad \bar{\psi}(0) = \bar{\psi}_0$

$$\rightsquigarrow \boxed{\begin{aligned} \bar{\psi}(t) &= \bar{\psi}_0 \exp\left\{ i \int_0^t d\tau\, V_2(x_B(t)) \right\} \\ \psi(t) &= \psi_0 \exp\left\{ -i \int_0^t d\tau\, V_2(x_B(t)) \right\} \end{aligned}}$$

Note that $V_2(x) = V_2(x_B) + V_2'(x_B)\, q\, \bar{\psi}_0\psi_0$

Homework: Conserved energy $\mathcal{E} = E + F\bar{\psi}_0\psi_0 \quad , \quad E, F \in \mathbb{R}$

$$\mathcal{E} := \frac{1}{2}\dot{x}^2 + V_1(x) + V_2(x)\bar{\psi}_0\psi_0 = \text{const.}$$

$\dot{x} = \dot{x}_B + \dot{q}\,\bar{\psi}_0\psi_0 \quad \rightsquigarrow \quad \frac{1}{2}\dot{x}^2 = \frac{1}{2}\dot{x}_B^2 + \dot{x}_B\dot{q}\,\bar{\psi}_0\psi_0$

$V_1(x) = V_1(x_B) + V_1'(x_B)\, q\, \bar{\psi}_0\psi_0$

$V_2(x)\bar{\psi}_0\psi_0 = V_2(x_B)\bar{\psi}_0\psi_0$

Compare body and soul

- $$\boxed{E = \frac{1}{2}\dot{x}_B^2 + V_1(x_B)} \iff \ddot{x}_B = -V_1'(x_B)$$

  integrable std. classical dynamics

- $$F = \dot{x}_B \dot{q} + V_1'(x_B)\, q + V_2(x_B)$$

Solution:

$$\boxed{q(t) = \dot{x}_B(t)\left[\frac{q(0)}{\dot{x}_B(0)} + \int_0^t d\tau\, \frac{F - V_2(x_B(\tau))}{\dot{x}_B^2(\tau)}\right]}$$

Proof:
$$\dot{q}(t) = \frac{\ddot{x}_B(t)}{\dot{x}_B(t)}\, q(t) + \dot{x}_B(t)\, \frac{F - V_2(x_B(t))}{\dot{x}_B^2(t)}$$

$$\dot{q}(t)\,\dot{x}_B(t) = \ddot{x}_B(t)\, q(t) + F - V_2(x_B(t))$$

$$= -V_1'(x_B(t))\, q(t) + F - V_2(x_B(t)) \quad \#$$

Conclusion:

1-dim. pseudo-classical mechanics is integrable

Solution is in essence characterised by solution of the std. classical x_B, the body of the bosonic degree of freedom