Exercise 12: The Correction Factor K_0 for Instantons

$$K_0 = \lim_{t\to\infty}\left[\frac{\det(-\partial_\tau^2+\omega^2)}{\det\left(-\partial_\tau^2+\frac{1}{m}V''[\bar{x}(\tau)]\right)}\right]^{1/2}$$

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$\sigma \to \tau$

$\tau \to t$ Notation

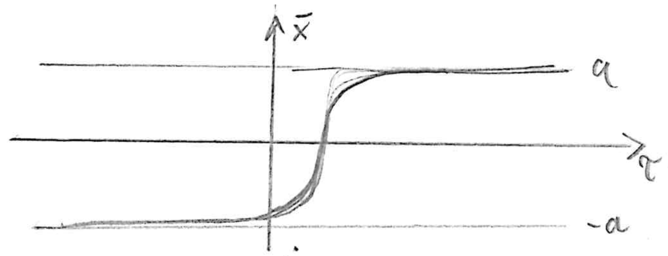
a) The zero-mode Problem

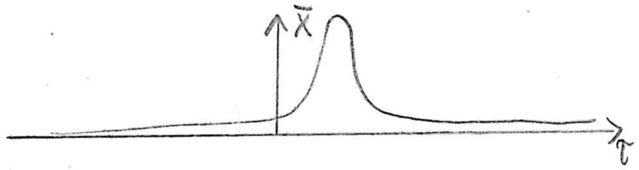
$$m\ddot{\bar{x}} = V'(\bar{x}) \rightsquigarrow m\dddot{\bar{x}} = V''(\bar{x})\dot{\bar{x}} \rightsquigarrow \left[-\partial_\tau^2+\tfrac{1}{m}V''(\bar{x})\right]\dot{\bar{x}}=0$$

$\dot{\bar{x}}$ is eigenfunction of fluctuation operator with eigenvalue 0

because $\dot{\bar{x}}(\pm t/2)\to 0$ for $t\to\infty$ (so obeys the boundary conditions)

Remember:





$\dot{\bar{x}}$ has No zero's
$\rightsquigarrow$ ground-state

Consider lowest eigenvalue for large t:

$$\lambda_0^{V''}(t) > 0 \quad \text{with} \quad \lim_{t\to\infty}\lambda_0^{V''}(t) = 0 \quad \text{and} \quad \varphi_0(\tau) \sim \dot{\bar{x}}(\tau)$$

zero-mode

However: $\lambda_n^{\omega^2}(t) = \omega^2 + \left(\frac{\pi n}{t}\right)^2 \longrightarrow \omega^2 > 0$ for $t\to\infty$

$\rightsquigarrow$ K_0 as defined above does not exist!

$\rightsquigarrow$ Correct definition to be found!

Problem is time-translation invariance:

With $\bar{x}_{t_1}(\tau)$ also $\bar{x}_{t_1+\varepsilon}(\tau) = \bar{x}_{t_1}(\tau - \varepsilon)$ is solution

$\leadsto$

$$\bar{x}_{t_1+\varepsilon}(\tau) \approx \bar{x}_{t_1}(\tau) - \varepsilon \dot{\bar{x}}_{t_1}(\tau) + O(\varepsilon^2)$$

fluctuation along the zero mode !

b) Resolving the zero-mode problem

Remember: $x(\tau) - \bar{x}(\tau) = q(\tau) = \sum_{n=0}^{\infty} c_n \varphi_n(\tau)$

eigenmodes of fluctuation operator

for large t: $\varphi_0(\tau) \approx N \dot{\bar{x}}_{t_1}(\tau)$

and because of $E = \frac{m}{2}\dot{\bar{x}}^2 - V(\bar{x}) \approx 0$

$\leadsto \quad \dot{\bar{x}}_{t_1}(\tau) = \sqrt{\frac{2V(\bar{x})}{m}}$

instanton action

Hence $\int_{-t/2}^{+t/2} d\tau\, \varphi_0^2(\tau) = N^2 \int_{-t/2}^{t/2} d\tau\, \dot{\bar{x}}_{t_1}^2(\tau) = N^2 \int_{-a}^{+a} dx \sqrt{\frac{2V(x)}{m}} = N^2 \frac{S_0}{m}$

$\leadsto \quad N = \sqrt{\frac{m}{S_0}}$

Result: $q(\tau) = c_0 \sqrt{\frac{m}{S_0}}\, \dot{\bar{x}}_{t_1}(\tau) + \sum_{n \neq 0} c_n \varphi_n(\tau)$

Consider fluctuation along zero-mode:

$$\delta q(\tau) = \delta c_0 \sqrt{\frac{m}{S_0}}\, \dot{\bar{x}}_{t_1}(\tau) = \delta c_0 \sqrt{\frac{m}{S_0}}\, \dot{\bar{x}}_0(\tau - t_1) = -\sqrt{\frac{m}{S_0}}\, \frac{\partial \bar{x}_{t_1}}{\partial t_1}\, \delta c_0$$

$$\stackrel{!}{=} \delta \bar{x}_{t_1}(\tau) = \frac{\partial \bar{x}_1(\tau)}{\partial t_1}\, \delta t_1$$

$$\Longrightarrow \delta t_1 = -\sqrt{\frac{m}{S_0}}\, \delta c_0$$

fluctuation = time-translation

Remember:

$$\int \mathcal{D}q[\tau] \exp\left\{-\frac{m}{2\hbar}\int_{-t_2}^{t_2} d\tau\, q(\tau)\, F\, q(\tau)\right\} = \mathcal{N} \int\int \prod_{n=0}^{\infty} \frac{dc_n}{(2\pi\hbar/m)^{1/2}}$$

$$\times \exp\left\{-\frac{m}{2\hbar}\sum_{n=0}^{\infty} \lambda_n c_n^2\right\}$$

$$= \mathcal{N} \int \frac{dc_0}{\sqrt{2\pi\hbar/m}} \prod_{n\neq 0}^{\infty} \int \frac{dc_n}{\sqrt{2\pi\hbar/m}} \exp\left\{-\frac{m}{2\hbar}\sum_{n\neq 0}^{\infty} \lambda_n c_n^2\right\}$$

as $\lambda_0 \searrow 0$ for $t \to \infty$

$$\text{But } \int \frac{dc_0}{\sqrt{2\pi\hbar/m}} = \int dt_1 \sqrt{\frac{S_0}{m}}\sqrt{\frac{m}{2\pi\hbar}} = \int dt_1 \sqrt{\frac{S_0}{2\pi\hbar}}$$

however the t_1-integration is already done in the instanton-counting !

$$= \mathcal{N} \sqrt{\frac{S_0}{2\pi\hbar}} \prod_{n\neq 0}^{\infty} \sqrt{\frac{1}{\lambda_n}}$$

Correct definition: $$\boxed{K_0 = \lim_{t\to\infty} \sqrt{\frac{S_0}{2\pi\hbar}} \left[\frac{\lambda_0(t)\, \det{}'(-\partial_\tau^2 + \omega^2)}{\det(-\partial_\tau^2 + \frac{1}{m}V''(\bar{x}))}\right]^{1/2}}$$

c) Calculation of K_0

Coleman: $$\frac{\lambda_0(t)\,\det(-\partial_\tau^2+\omega^2)}{\det(-\partial_\tau^2+\frac{1}{m}V''(\bar{x}))} = \frac{\lambda_0(t)\,\frac{1}{\omega}\sinh\omega t}{f_0(t/2)}$$

where $f_0(-\frac{t}{2})=0$, $\dot{f}_0(-\frac{t}{2})=1$ and

$$\left(-\partial_\tau^2+\frac{1}{m}V''(\bar{x}(\tau))\right)f_0(\tau)=0$$

$\leadsto$ $$K_0=\lim_{t\to\infty}\sqrt{\frac{S_0}{2\pi\hbar}}\sqrt{\frac{\lambda_0(t)\,e^{\omega t}}{2\omega\,f_0(t/2)}}$$

i) Calculation of f_0:

- Zero mode $\varphi_0(\tau)=\sqrt{\frac{m}{S_0}}\,\dot{\bar{x}}_0(\tau)=\varphi_0(-\tau)$ is solution symmetric

  but with wrong initial conditions

  as $\dot{\bar{x}}_0(\tau)\sim e^{-\omega|\tau|}$ for $|\tau|\to\infty$ (Example $\bar{x}(\tau)=a\tanh\frac{\omega\tau}{2}$)

Hence: $\varphi_0(\tau)\approx a_0\,e^{-\omega|\tau|}$ with $a_0:=\lim_{|\tau|\to\infty}e^{\omega|\tau|}\varphi_0(\tau)$

- linear independent solution:

$$\psi_0(\tau):=\varphi_0(\tau)\int_0^\tau d\tau'\,\frac{1}{\varphi_0^2(\tau')}=-\psi_0(-\tau)\quad\text{antisymmetric}$$

$\leadsto$ $$\dot{\psi}_0=\dot{\varphi}_0\int_0^\tau\frac{d\tau'}{\varphi_0^2(\tau')}+\varphi_0(\tau)\frac{1}{\varphi_0^2(\tau)}$$

multiply with φ_0: $$\varphi_0\dot{\psi}_0=\dot{\varphi}_0\psi_0+1$$

or $$\varphi_0\dot{\psi}_0-\dot{\varphi}_0\psi_0=1\quad\text{Wronski determinant}$$

apply ∂_τ: $\quad \cancel{\dot{\varphi}_0 \dot{\nu}_0} + \varphi_0 \ddot{\nu}_0 - \ddot{\varphi}_0 \nu_0 - \cancel{\dot{\varphi}_0 \dot{\nu}_0} = 0 \quad$ and $\ddot{\varphi}_0 = \sqrt{\frac{m}{S_0}}\, \dddot{x}(\tau)$

with $\dddot{x} = \frac{1}{m} V'' \dot{x} \quad \rightsquigarrow \quad \ddot{\varphi}_0 = \frac{1}{m} V''(\bar{x}) \varphi_0$

$$\rightsquigarrow \quad \varphi_0 \left(-\partial_\tau + \tfrac{1}{m} V''(\bar{x})\right) \nu_0(\tau) = 0$$

With $\nu_0(t) \xrightarrow[t\to\infty]{} a_0 e^{-\omega t} \int_0^t \frac{d\tau}{a_0^2 e^{-2\omega\tau}} \sim \frac{e^{\omega t}}{2a_0\omega}$

Hence: $\quad \nu_0(t) \approx \pm \frac{e^{\omega|t|}}{2a_0\omega}$ for $t \to \pm\infty$

Lemma: $f_0(\tau) = \varphi_0(-\frac{t}{2})\nu_0(\tau) - \nu_0(-\frac{t}{2})\varphi_0(\tau)$

Proof: $\quad f_0(-t/2) = 0$

$\dot{f}_0(-t/2) = \varphi_0(-t/2)\dot{\nu}_0(-t/2) - \nu_0(-t/2)\dot{\varphi}_0(-t/2) = 1$ Wronski det.

Furthermore: $f_0(t/2) \approx a_0 e^{-\frac{\omega t}{2}} \frac{e^{\frac{\omega t}{2}}}{2a_0\omega} \cdot 2 = \frac{1}{\omega}$

$$\boxed{\lim_{t\to\infty} f_0(t/2) = \frac{1}{\omega}}$$

ii) Calculate $\lambda_0(\tau)$

Consider $\left(-\partial_\tau^2 + \frac{1}{m} V''(\bar{x})\right) f_\lambda(\tau) = \lambda f_\lambda(\tau)$

$\lambda = \lambda(t)$ Eigenvalue $\iff f_\lambda(t/2) = 0$

with $f_\lambda(-t/2) = 0$ and $\dot{f}_\lambda(-t/2) = 1$

Let $g(\tau) := \frac{\partial}{\partial \lambda} f_\lambda(\tau)\Big|_{\lambda=0}$

obviously $g(-t/2) = 0 = \dot{g}(-t/2)$

and $\left(-\partial_\tau^2 + \frac{1}{m} V''(\bar{x})\right) g(\tau) = f_0(\tau)$ ⊛

Solution:
$$g(\tau) = \int_{-t/2}^{\tau} d\tau' \left[\psi_0(\tau')\,\varphi_0(\tau) - \psi_0(\tau)\,\varphi_0(\tau')\right] f_0(\tau')$$

$$\dot{g}(\tau) = \dot{\varphi}_0(\tau) \int_{-t/2}^{\tau} d\tau'\, \psi_0(\tau') f_0(\tau') + \cancel{\psi_0(\tau)\varphi_0(\tau) f_0(\tau)} - \dot{\psi}_0(\tau) \int_{-t/2}^{\tau} d\tau'\, \varphi_0(\tau') f_0(\tau') - \cancel{\psi_0(\tau)\varphi_0(\tau) f_0(\tau)}$$

$$\ddot{g}(\tau) = \ddot{\varphi}_0(\tau) \int_{-t/2}^{\tau} d\tau'\, \psi_0(\tau') f_0(\tau') + \dot{\varphi}_0 \psi_0 f_0 - \ddot{\psi}_0 \int_{-t/2}^{\tau} d\tau'\, \varphi_0(\tau') f_0(\tau') - \dot{\psi}_0 \varphi_0 f_0$$

$\frac{1}{m} V''(\bar{x})\, \varphi_0 / \psi_0$ $\qquad$ $f_0(\dot{\varphi}_0 \psi_0 - \dot{\psi}_0 \varphi_0) = -f_0$

$$= -f_0 + \frac{1}{m} V''(\bar{x}) \int_{-t/2}^{\tau} d\tau' \left[\psi_0(\tau')\varphi_0(\tau) f_0(\tau') - \psi_0(\tau)\varphi_0(\tau') f_0(\tau')\right]$$

$$= -f_0 + \frac{1}{m} V''(\bar{x})\, g(\tau)$$

$$\Rightarrow \left(-\partial_\tau^2 + \frac{1}{m} V''(\bar{x})\right) g(\tau) = f_0(\tau) \qquad \blacksquare$$

Approximation for $\lambda \ll 1$:

$$f_\lambda(\tau) \approx f_0(\tau) + \lambda\, g(\tau)$$

$$f_\lambda(\tfrac{t}{2}) \approx f_0(t/2) + \lambda \int_{-t/2}^{+t/2} d\tau \left[\psi_0(\tau)\varphi_0(\tfrac{t}{2}) - \psi_0(\tfrac{t}{2})\varphi_0(\tau)\right] f_0(\tau)$$

Lemma

$\varphi_0(\tfrac{t}{2})\psi_0(\tau) - \psi_0(-\tfrac{t}{2})\varphi_0(\tau)$

sym. anti-sym.

$$= f_0(\tfrac{t}{2}) + \lambda \int_{-t/2}^{t/2} d\tau \left[\psi_0^2(\tau)\varphi_0^2(\tfrac{t}{2}) - \psi_0^2(\tfrac{t}{2})\varphi_0^2(\tau)\right]$$

largest t behaviour:

$$f_0(t/2) \longrightarrow \frac{1}{\omega}$$

$$\varphi_0^2(t/2)\int_{-t/2}^{t/2} d\tau\, \psi_0^2(\tau) \sim a_0^2 e^{-\omega t}\, 2\int_0^{t/2} d\tau \underbrace{\psi_0^2(\tau)}_{\sim e^{2\omega\tau}} \sim 2a_0^2 = O(1)$$

$$\psi_0^2(t/2)\underbrace{\int_{-t/2}^{t/2} d\tau\, \varphi_0^2(\tau)}_{=1 \text{ Normalized}} \sim \frac{1}{4a_0^2\omega^2}e^{\omega t}$$

$$\rightsquigarrow\quad f_\lambda(\tfrac{t}{2}) \approx \frac{1}{\omega} + \lambda\left(O(1) - \frac{1}{4a_0^2\omega^2}e^{\omega t}\right) \overset{!}{=} 0 \quad \text{Eigenvalue Condition!}$$

$$\rightsquigarrow\quad \frac{1}{\omega} - \lambda\frac{e^{\omega t}}{4a_0^2\omega^2} = 0 \quad\rightsquigarrow\quad \boxed{\lambda_0(t) \approx 4a_0^2\omega e^{-\omega t}} \quad \text{for } t\to\infty$$

Remember: $K_0 = \lim_{t\to\infty}\sqrt{\frac{S_0}{2\pi\hbar}}\sqrt{\frac{\lambda_0(t)e^{\omega t}}{2\omega f_0(t/2)}}$

$$\boxed{K_0 = \sqrt{\frac{S_0}{2\pi\hbar}}\sqrt{2a_0^2\omega} = a_0\sqrt{\frac{S_0\omega}{\pi\hbar}} = \sqrt{\frac{m\omega}{\pi\hbar}}\lim_{t\to\infty} e^{\omega t}\dot{x}_0(t)}$$

$S_0 = \int_{-a}^{+a} dx\sqrt{2mV(x)}$, $a_0 = \lim_{t\to\infty} e^{\omega t}\varphi_0(t) = \sqrt{\frac{m}{S_0}}\lim_{t\to\infty} e^{\omega t}\dot{x}_0(t)$

$$\Delta E = E_1 - E_0 = 2\hbar K_0 e^{-S_0/\hbar} = 2\sqrt{\frac{m\hbar\omega}{\pi}}\, e^{-S_0/\hbar} \lim_{t\to\infty} e^{\omega t}\dot{\bar{x}}(t)$$

$$= 2a_0\sqrt{\frac{S_0\hbar\omega}{\pi}}\, e^{-S_0/\hbar}$$

Example: $V(x) = \frac{m\omega^2}{8a^2}(x^2-a^2)^2$

$\rightsquigarrow\ S_0 = \frac{2}{3}m\omega a^2,\quad \bar{x}_0(t) = a\tanh\frac{\omega t}{2},\quad \dot{\bar{x}}_0(t) = \frac{a\omega}{2}\frac{1}{\cosh^2\frac{\omega t}{2}}$

$\dot{\bar{x}}_0(t) \sim 2a\omega e^{-\omega t} \rightsquigarrow a_0 = 2a\omega$