Exercise 1 : The generalised Lie-Trotter Formula

Note: $\frac{z}{1+z} \leq \ln(1+z) \leq z$ for $z > -1$

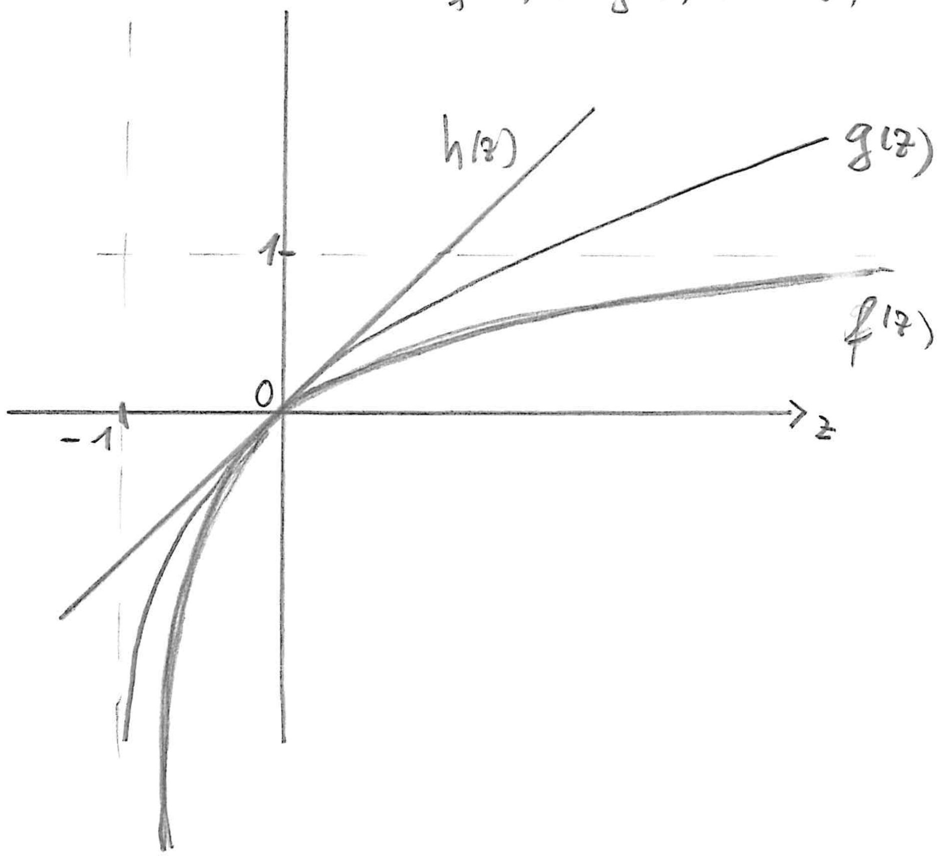
Obvious as $f(z) := \frac{z}{1+z} \rightsquigarrow f'(z) = \frac{1}{(1+z)^2}$

$g(z) := \ln(1+z) \rightsquigarrow g'(z) = \frac{1}{1+z}$

$h(z) := z \rightsquigarrow h'(z) = 1$

$h(0) = g(0) = f(0) = 0$ and $f'(z) < g'(z) < h'(z)$ for $z > 0$

$f'(z) > g'(z) > h'(z)$ for $z < 0$



Lemma: Let y_n be series such that $y_n \to 0$ for $n \to \infty$

Then $\left(1 + \frac{x + y_n}{n}\right)^n \to e^x$ for $n \to \infty$

Proof: Use Note with $z = \frac{x+y_n}{n}$

$$\frac{x+y_n}{1+\frac{x+y_n}{n}} \le n \ln\left(1+\frac{x+y_n}{n}\right) \le x + y_n$$

$$\frac{x+y_n}{1+\frac{x+y_n}{n}} \to x, \qquad x + y_n \to x \quad \text{for } n \to \infty$$

$$\rightsquigarrow \exp\left\{n \ln\left(1+\frac{x+y_n}{n}\right)\right\} = \left(1+\frac{x+y_n}{n}\right)^n \longrightarrow e^x$$

Theorem: Let $\{\hat{F}(s)\}_{s\geq 0}$ be 1-parameter set of bounded operators on $\mathcal{H}$ with $\hat{F}(0) = \mathbb{1}$ and $\hat{F}'(0) = \lim_{s \searrow 0} \frac{\hat{F}(s) - \mathbb{1}}{s}$

then

$$\underset{N\to\infty}{\text{s-lim}} \left[\hat{F}\left(\tfrac{t}{N}\right)\right]^N = e^{t \hat{F}'(0)}$$

generalised Lie-Trotter formula

In particular for $\hat{F}(s) = e^{s\hat{T}} e^{s\hat{V}} \rightsquigarrow \hat{F}(0) = \mathbb{1}$

$\hat{F}'(s) = \hat{T}\hat{F}(s) + \hat{F}(s)\hat{V} \rightsquigarrow \hat{F}'(0) = \hat{T} + \hat{V}$

$$\Rightarrow \quad \underset{N\to\infty}{\text{s-lim}} \left[e^{t\hat{T}/N} e^{t\hat{V}/N}\right]^N = e^{t(\hat{T}+\hat{V})}$$

<u>Proof:</u> • $\hat{F}(s) = 1 + s\hat{F}'(0) + o(s)$, $\lim_{s \searrow 0} \frac{o(s)}{s} = 0$

• $\hat{F}(t/N) = 1 + \frac{1}{N}\left(t\hat{F}'(0) + \hat{R}_N(t)\right)$, $\lim_{N \to \infty} \hat{R}_N(t) = 0$

$$\Rightarrow \left[\hat{F}(t/N)\right]^N = \left[1 + \frac{1}{N}\left(t\hat{F}'(0) + \hat{R}_N(t)\right)\right]^N \xrightarrow[N \to \infty]{} e^{t\hat{F}'(0)}$$

Remarks: Proof for $\dim \mathcal{H} < \infty$ but also valid for $\dim \mathcal{H} = \infty$

Let T, V and $H := T + V$ be bounded from below and essentially self-adjoint on $\mathcal{H}$ $\Rightarrow$ $e^{-H} = \text{s-}\lim_{n \to \infty} \left(e^{-T/n} e^{-V/n}\right)^n$

For proof see Glimm / Jaffe

<u>Lie-Trotter formula:</u> Let T, V and $T + V$ be self-adjoint on $\mathcal{H}$

$$e^{-it(T+V)} = \text{s-}\lim_{N \to \infty} \left[e^{-itT/N} e^{-itV/N}\right]^N$$

Notes:

strong limit: $A = \text{s-}\lim_{n \to \infty} A_n :\Leftrightarrow \| A_n|\psi\rangle - A|\psi\rangle \| \xrightarrow[n \to \infty]{} 0 \ \forall |\psi\rangle \in \mathcal{H}$

norm limit: $A = \text{n-}\lim_{n \to \infty} A_n :\Leftrightarrow \| A_n - A \| \xrightarrow[n \to \infty]{} 0$

weak limit: $A = \text{w-}\lim_{n \to \infty} A_n :\Leftrightarrow \langle (A - A_n)\psi | \varphi \rangle \xrightarrow[n \to \infty]{} 0$

$\forall \psi, \varphi \in \mathcal{H}$

Exercise 2 : The free particle propagator

$$K_0(x'',x',t) := \langle x''| e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t} |x'\rangle$$

$\uparrow$ $1 = \int dp\, |p\rangle\langle p|$

$$= \int dp\, e^{-\frac{ip^2t}{2m\hbar}} \langle x''|p\rangle\langle p|x'\rangle$$

use $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar}px}$

$$= \frac{1}{2\pi\hbar}\int dp\, \exp\left\{-\frac{it}{2m\hbar}p^2 + \frac{i}{\hbar}(x''-x')p\right\}$$

use Homework 1b with $a = \frac{it}{2m\hbar}$, $b = \frac{i}{\hbar}(x''-x')$

$$= \sqrt{\frac{\pi\, 2m\hbar}{it}}\, \exp\left\{-\frac{(x''-x')^2}{\hbar^2}\,\frac{2m\hbar}{4it}\right\}\frac{1}{2\pi\hbar}$$

$$= \sqrt{\frac{m}{2\pi i\hbar t}}\, \exp\left\{\frac{i}{\hbar}\,\frac{m}{2}\,\frac{(x''-x')^2}{t}\right\}$$

$\sim$

$\underbrace{\langle x|P|p\rangle = p\langle x|p\rangle = \frac{\hbar}{i}\partial_x \langle x|p\rangle}$

$\Rightarrow \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar}px}$

Exercise 3: The harmonic oscillator

Lagrangian: $L(\dot{x},x) = \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2 \quad , \quad \omega > 0$

Classical path: $x_{cl}(\tau) = a\cos\omega\tau + b\sin\omega\tau$

with $x_{cl}(0) = x'$, $x_{cl}(t) = x''$ $\Rightarrow$ a and b

Classical action: Homework

$$S_{cl}(x'',x',t) = \frac{m\omega}{2\sin\omega t}\left[(x''^2 + x'^2)\cos\omega t - 2x''x'\right]$$

Propagator:

$$K_{\omega^2}(x'',x';t) = \int_{x(0)=x'}^{x(t)=x''} \mathcal{D}[x(\tau)] \exp\left\{\frac{i}{\hbar}\int_0^t d\tau\, \frac{m}{2}(\dot{x}^2 - \omega^2 x^2)\right\}$$

$$\overset{x = x_{cl}+q}{=} e^{\frac{i}{\hbar}S_{cl}} \int_{q(0)=0}^{q(t)=0} \mathcal{D}[q(\tau)] \exp\left\{\frac{i}{\hbar}\underbrace{\int_0^t d\tau\, \frac{m}{2}(\dot{q}^2 - \omega q^2)}_{\delta^2 S}\right\}$$

$$=: e^{\frac{i}{\hbar}S_{cl}}\, F_{\omega^2}(t)$$

Fourier expansion:

$$q(\tau) = \sum_k a_k \sin\frac{k\pi\tau}{t} \quad , \quad k = 1,2,3,\dots,N-1$$

Note $k = N\Delta$ $\frac{t}{N} = \varepsilon$

Change of variables:

$$\int dq_1 \cdots \int dq_{N-1} = J(t) \int da_1 \cdots \int da_{N-1}$$

Linear transformation $\rightarrow$ Jacobian $J(t)$ is constant!

Consider action:

$$\delta^2 S = \int_0^t d\tau \, \frac{m}{2} \left(\dot{q}^2 - \omega^2 q^2 \right)$$

$$= \sum_{k,k'} \frac{m}{2} a_k a_{k'} \int_0^t d\tau \left[\frac{k k' \pi^2}{t^2} \underbrace{\cos\frac{k\pi\tau}{t} \cos\frac{k'\pi\tau}{t}}_{\rightarrow \frac{t}{2}\delta_{kk'}} - \omega^2 \underbrace{\sin\frac{k\pi\tau}{t} \sin\frac{k'\pi\tau}{t}}_{\rightarrow \frac{t}{2}\delta_{kk'}} \right]$$

$$= \sum_k \frac{mt}{4} a_k^2 \left[\frac{k^2\pi^2}{t^2} - \omega^2 \right]$$

$$= \sum_k \frac{m k^2 \pi^2}{4t} \left[1 - \left(\frac{\omega t}{k\pi} \right)^2 \right]$$

Fresnel integral: assume $\omega t < \pi$ (Homework

$$\int_{-\infty}^{+\infty} da_k \, e^{i \frac{m k^2 \pi^2}{4\hbar t} \overbrace{\left[1 - \frac{\omega^2 t^2}{k^2\pi^2} \right]}^{>0 \; \forall k} a_k^2} = \frac{c(t)}{\left(1 - \frac{\omega^2 t^2}{k^2 \pi^2} \right)^{1/2}}$$

Euler product formula: allows limit $N \rightarrow \infty$

$$\prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2} \right) = \frac{\sin x}{x}$$

Result:

$$\tilde{F}_{\omega^2}(t) = \underbrace{J(t)\,[c(t)]^{\infty}}_{\text{meaning?}} \sqrt{\frac{\omega t}{\sin \omega t}}$$

Consider: $\omega = 0$

we know $\tilde{F}_0(t) = \sqrt{\frac{m}{2\pi i \hbar t}}$

$$\rightsquigarrow \quad F_{\omega^2}(t) = \sqrt{\frac{m\omega}{2\pi i \hbar \sin \omega t}}$$

Propagator:

$$K_{\omega^2}(x'', x'; t) = \sqrt{\frac{m\omega}{2\pi i \hbar \sin \omega t}}\; e^{\frac{i}{\hbar}\frac{m\omega}{2\sin \omega t}\left[(x''^2 + x'^2)\cos \omega t - 2x''x'\right]}$$

Comment:

- Above result only valid for $\omega t < \pi$
- Singularities (caustics) occur at $\omega t = n\pi$
- Alternative calculation via Coleman theorem

$$K_{\omega^2}(x'', x'; t) = \sqrt{\frac{m\omega}{2\pi i \hbar |\sin \omega t|}}\; e^{-i\frac{\pi}{2}\left[\frac{\omega t}{\pi}\right]}\; e^{\frac{i}{\hbar} S_{cl}} \quad \text{for} \quad \omega t \neq \pi n$$

with $[z] := \max\limits_{n \in \mathbb{N}} n < z$ Morse index

$$K_{\omega^2}(x'',x';t) = e^{-i\frac{\pi}{2}n}\,\delta(x''-(-1)^n x') \qquad t = \frac{\pi n}{\omega}$$

↗ B. Felsager, "Geometry, Particles and Fields"

Van Vleck - Pauli - Morette:

Consider $\dfrac{\partial^2 S_{cl}}{\partial x''\partial x'} = -\dfrac{m\omega}{\sin\omega t}$ again $\omega t < \pi$

$$\rightsquigarrow \boxed{K_{\omega^2}(x'',x';t) = \sqrt{\frac{i}{2\pi\hbar}\,\frac{\partial^2 S_{cl}}{\partial x''\partial x'}}\; e^{\frac{i}{\hbar}S_{cl}}}$$