Lecture 6

23 Stochastic processes

23.1 Some basics from probability theory

Probability space:
Let

Ω: Sample space (set of events)
$\mathcal{A}$: σ-algebra of Ω
P: Probability measure $P : \mathcal{A} \to [0,1]$

The set $(\Omega, \mathcal{A}, P)$ is called *probability space*
$\omega \in \Omega$ is called elementary event
$E \in \mathcal{A}$ is called *event*
$P(E)$ is the probability of event E

Conditions on probability measure:

(i) $0 \leq P(E) \leq 1 \qquad \forall\, E \in \mathcal{A}$

(ii) $P(\Omega) = 1$

(iii) Let $E_1, E_2, \ldots$ be disjoint events $\quad \Leftrightarrow \quad E_i \cap E_j = \emptyset \;\; \forall\, i \neq j$

(iv) $\overline{E} := \Omega \backslash E$ "not E" event $\implies P(\overline{E}) = 1 - P(E)$

(v) $P(\emptyset) = 0 \qquad \emptyset =$ impossible event

Random variable:
A measurable function

$$X : \Omega \to \Omega'$$

is called *random variable*. It induces a probability measure (push forward measure)

$$P_X(E') = P(\{\omega \in \Omega | X(\omega) \in E'\}\,, \qquad \forall\, E' \in \Omega'\,.$$

P_X is called *distribution* of X.

23.2 Stochastic processes

Let T be an index set with $\tau \in T$ and

$$X_\tau : \begin{array}{l} \Omega \to \mathbb{R}^d = \Omega' \\ \omega \mapsto X_\tau(\omega) \end{array} \qquad \forall\, \tau \in T$$

be a family of $\mathbb{R}^d$-valued random variables with $(\omega, \mathcal{A}, P)$ as probability space.
Then $(\omega, \mathcal{A}, P, \{X_\tau\}_{\tau \in T})$ is called a *stochastic process* with index set T and state (or sample) space $\mathbb{R}^d$.

Example: Brownian motion
Here $\Omega = \Omega' = \mathbb{R}^d$
$\omega \in \Omega$ denotes an elementary path taken by the particle during time $0 \leq \tau \leq T$.
$X_\tau(\omega) = \omega(\tau)$ position of Brownian particle at time $\tau \quad = \quad$ random variable for each τ
$X_\cdot(\omega)$ for fixed $\omega \in \Omega$ is called *path* or *realisation* of the stochastic process

23.3 Finite-dimensional marginal distribution

The joint probability density

$$P_n(x_n t_n, x_{n-1} t_{n-1}, \ldots, x_1 t_1) := \big\langle \delta(X_{t_n} - x_n)\delta(X_{t_{n-1}} - x_{n-1}) \cdots \delta(X_{t_1}) - x_1) \big\rangle_{P_{X_\tau}}$$

of the random variables X_{t_i} with $t_i \in T$, $i = 1, 2, 3, \ldots, n$, $n \in \mathbb{N}$ are called *marginal distributions*.

Properties:
- Positivity: $P_n(x_n t_n, x_{n-1} t_{n-1}, \ldots, x_1 t_1) \geq 0$
- Symmetry: $P_n(x_{\pi(n)} t_{\pi(n)}, x_{\pi(n-1)} t_{\pi(n-1)}, \ldots, x_{\pi(1)} t_{\pi(1)}) = P_n(x_n t_n, x_{n-1} t_{n-1}, \ldots, x_1 t_1)$
  for all permutations $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}$
- Compatability: $\int_{\mathbb{R}^d} \mathrm{d}^d x_n \, P_n(x_n t_n, x_{n-1} t_{n-1}, \ldots, x_1 t_1) = P_{n-1}(x_{n-1} t_{n-1}, \ldots, x_1 t_1)$
- Normalisation: $\int_{\mathbb{R}^d} \mathrm{d}^d x_1 \, P_1(x_1 t_1) = 1$

Kolmogorov's fundamental (or extension) theorem:
For each family $\{P_n(\cdots)\}_{n\in\mathbb{N}}$ of finite dimensional distributions, obeying above properties, exists a stochastic process $(\omega, \mathcal{A}, P, \{X_\tau\}_{\tau\in T})$ with these marginal distributions. The stochastic process is in essence unique.
In other words, a stochastic process is in essence uniquely defined via its family of marginal distributions.

23.4 Brownian motion

1827 Robert Brown: Observed irregular motion of pollen on a static liquid originating from random hits of pollen by molecules of the liquid.

1905 Albert Einstein: Provided first correct theoretical description:

- Assume motions within two successive time intervals τ (small) are independent (no correlation of hits)
- Within τ position x varies by amount Δ
- Δ is a random variable with a probability distribution $\Phi(\Delta)$ such that $\Phi(-\Delta) = \Phi(\Delta)$ and $\langle \Delta^2 \rangle \sim \tau$.

Let $f(x,t)$ be the probability distribution density to find particle at position x at time t. Then

$$f(x, t+\tau) = \int \mathrm{d}\Delta \, \Phi(\Delta) f(x+\Delta, t)$$

and for small τ and Δ we have

$$f(x, t+\tau) \approx f(x,t) + \tau \dot{f}(x,t)$$
$$\int \mathrm{d}\Delta \, \Phi(\Delta) f(x+\Delta, t) \approx f(x,t) \underbrace{\int \mathrm{d}\Delta \, \Phi(\Delta)}_{=1} + f'(x,t) \underbrace{\int \mathrm{d}\Delta \, \Delta\Phi(\Delta)}_{=0} + f''(x,t) \underbrace{\int \mathrm{d}\Delta \, \tfrac{1}{2}\Delta^2 \Phi(\Delta)}_{=:D\tau}$$

Comparing both sides results in the diffusion equation

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial^2 x}$$

Einstein: $D = \frac{k_B T}{6\pi\eta a}$ with T temperature, η viscosity and a radius of pollen.
Obviously $\langle \Delta^2 \rangle = 2D\tau$ and with initial condition $f(x,0) = \delta(x)$ the solution of the diffusion equation is given by the Gaussian distribution

$$f(x,t) = \frac{1}{\sqrt{4\pi D\tau}} \, \mathrm{e}^{-\frac{x^2}{4D\tau}}$$

characterising the probability distribution of X_t.

1907 Paul Langevin: Starts with Newton's eq. for X_t

$$m\ddot{X}_t + \underbrace{6\pi\eta a \dot{X}_t}_{=\text{friction}} = F \quad \leftarrow \quad \text{random force}$$

Multiply by X_t

$$m\ddot{X}_t X_t + 6\pi\eta a \dot{X}_t X_t = F X_t$$

Formally we have $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} X_t^2 = \dot{X}_t^2 + \ddot{X}_t X_t$ resulting in

$$\frac{m}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} X_t^2 - m\dot{X}_t^2 + 3\pi\eta a \frac{\mathrm{d}}{\mathrm{d}t} X_t^2 = F X_t$$

Now we take the mean value, assume thermodyn. equilibrium $\frac{m}{2}\langle \dot{X}_t^2 \rangle = \frac{1}{2} k_B T$ (1-dim.) and set $x^2(t) := \langle X_t^2 \rangle$

$$\frac{m}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} x^2(t) - k_B T + 3\pi\eta a \frac{\mathrm{d}}{\mathrm{d}t} x^2(t) = \langle F X_t \rangle \overset{\text{independent}}{=} \langle F \rangle \underbrace{\langle X_t \rangle}_{=0} = 0$$

Solution is same as Einstein's: $\quad x^2(t) = \langle X_t^2 \rangle = \frac{k_B T}{3\pi\eta a} t = 2Dt$

23.5 Stationary Markov processes

Conditional probability density: Let $\tau_1 \le \tau_2 \le \cdots \le \tau_m \le t_1 \le \cdots \le t_n$ then

$$P_{n|m}(x_n t_n, \cdots, x_1 t_1 | \xi_m \tau_m, \cdots, \xi_1 \tau_1) := \frac{P_{n+m}(x_n t_n, \cdots, x_n t_n, \xi_m \tau_m, \cdots, \xi_1 \tau_1)}{P_m(\xi_m \tau_m, \cdots, \xi_1 \tau_1)}$$

is the *conditional probability density* for X_t taking values x_i at times t_i under condition that the values ξ_j at τ_j were realised by X_t.
Obvious relations:

$$P_1(x_2 t_2) = \int \mathrm{d}x_1 \, P_2(x_2 t_2, x_1 t_1) = \int \mathrm{d}x_1 \, P_{1|1}(x_2 t_2 | x_1 t_1) P_1(x_1 t_1)$$

$$P_{1|1}(x_3 t_3 | x_1 t_1) = \int \mathrm{d}x_2 \, P_{2|1}(x_3 t_3, x_2 t_2 | x_1 t_1) = \int \mathrm{d}x_2 \, P_{1|2}(x_3 t_3 | x_2 t_2, x_1 t_1) P_{1|1}(x_2 t_2 | x_1 t_1)$$

Markov assumption: If

$$P_{n|m}(x_n t_n, \cdots, x_n t_n | \xi_m \tau_m, \cdots, \xi_1 \tau_1) = P_{n|1}(x_n t_n, \cdots, x_n t_n | \xi_m \tau_m)$$

depends only on the **last** condition $\iff$: **Markov process**

Chapman-Kolmogorov equation:

$$\boxed{P_{1|1}(x_3 t_3 | x_1 t_1) = \int \mathrm{d}x_2 \, P_{1|1}(x_3 t_3 | x_2 t_2) P_{1|1}(x_2 t_2 | x_1 t_1)}$$

Note:

$$P_n(x_n t_n, x_{n-1} t_{n-1}, \ldots, x_1 t_1) = P_{1|1}(x_n t_n | x_{n-1} t_{n-1}) P_{1|1}(x_{n-1} t_{n-1} | x_{n-2} t_{n-2}) \cdots P_{1|1}(x_2 t_2 | x_1 t_1)$$

$P_{1|1}$ is called the *transition probability density* of the Markov process.
If it depends only of the time difference then it is called a *stationary Markov process* with

$$m_{t_2-t_1}(x_2, x_1) := P_{1|1}(x_2t_2|x_1t_1)$$

Stationary Markov process: Is uniquely defined by

- State space: $Q \subseteq \mathbb{R}^d$
- Initial distribution: $P_0(x)$, $x \in Q$, at $t = 0$
- Transition probability density: $m_t(x_2, x_1)$, $t > 0$, obeying below requirements
  - $m_t(x_2, x_1) \geq 0$
  - $\int_Q \mathrm{d}x_2\, m_t(x_2, x_1) = 1$
  - $\lim_{t\searrow 0} m_t(x_2, x_1) = \delta(x_2 - x_1)$
  - $\int_Q \mathrm{d}x_2\, m_t(x_3, x_2) m_t(x_2, x_1) = m_t(x_3, x_1)$
- If, in addition, the *Lindeberg condition* holds, i.e.,

$$\lim_{t\searrow 0} \frac{1}{t} \int_{|x_2-x_1|>\varepsilon} \mathrm{d}x_2\, m_t(x_2, x_1) = 0 \qquad \forall\, \varepsilon > 0\, , x_1 \in Q$$

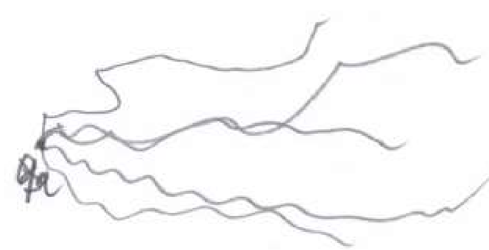
  then this is a *continuous Markov process* who's paths (realisations) are almost sure continuous (no jumps)

Realisations of a continuous stationary markov process, with initial distribution $P_0(x) = \delta(x - q_a)$ are all continuous open-end paths living in Q and starting in $q_a \in Q$.
We denote this set of paths by the *path space* $\mathcal{C}(Q, q_a)$.
From now on we use the notation $M_t(\omega) \equiv X_t(\omega) \in \mathcal{C}(Q, q_a)$ for such Markov prcesses.

23.6 Path integral representation of stationary Markov processes

Ingredients:

- Continuous stationary markov process $M_t(\omega)$, $\omega \in \Omega$, $t \geq 0$
- State (or confinguration) space $Q \subseteq \mathbb{R}^d$
- Initial distribution $P_0(q) = \delta(q - q_a)$
- Path space $\mathcal{C}(Q, q_a)$, all open-end paths $M_.(\omega)$ starting at q_a



Path: for fixed $\omega \in \Omega$ with $M_0(\omega) = q_a$

$$x : \begin{array}{ccc} \mathbb{R}_+^0 & \to & Q \\ t & \mapsto & M_t(\omega) \end{array}$$

Probability measure: P

- Initial condition: $P(\{\omega \in \Omega | M_0(\omega) = q_a\}) = 1$
- Continuity: $P(\{\omega \in \Omega | \lim_{t' \to t} M_{t'}(\omega) = M_t(\omega)\}) = 1$
- Marginal distribution:

  $$P(\{\omega \in \Omega | M_{t_n}(\omega) = q_n, \dots , M_{t_1}(\omega) = q_1\}) = m_{t_n - t_{n-1}}(q_n, q_{n-1}) \cdots m_{t_1 - t_0}(q_1, q_a)$$

  with
  $$0 = t_0 < t_1 < \cdots < t_n \, , \qquad q_i \in Q \, , \quad i = 1, 2, \dots , n \, , \qquad \forall \, n \in \mathbb{N} \, .$$

- Induced mapping:
  $$\phi : \begin{array}{ccc} \Omega & \to & \mathcal{C}(Q, q_a) \\ \omega & \mapsto & x \quad \text{Realisation} \end{array}$$

  induces a probability measure on $\mathcal{C}(Q, q_a)$: $M := P \circ \phi^{-1}$

The M-path integral:
Consider arbitrary real-valued functional F

$$F : \begin{array}{ccc} \mathcal{C}(Q, q_a) & \to & \mathbb{R} \\ x & \mapsto & F[x] \end{array}$$

Then the expectation value is given by the *M-path integral*

$$\boxed{\langle F[M_t] \rangle = \int_{\mathcal{C}(Q,q_a)} \mathrm{d}M[x] \, F[x]}$$

The measure can explicitly be defined via so-called cylinder functionals

$$F[x] = f_n(x(t_n), \dots , x(t_1)) \qquad 0 < t_1 < \cdots < t_n \, , \qquad n \in \mathbb{N}$$

as follows

$$\begin{aligned} &\int_{\mathcal{C}(Q,q_a)} \mathrm{d}M[x] \, f_n(x(t_n), \dots , x(t_1)) \\ &\qquad := \int_Q \mathrm{d}q_n \cdots \int_Q \mathrm{d}q_1 \, m_{t_n - t_{n-1}}(q_n, q_{n-1}) \cdots m_{t_1 - t_0}(q_1, q_a) \, f_n(q_n, \dots . q_1) \end{aligned}$$

This is similar to the definition of a Riemann or Lebesque integral.
Extension to general functional is then via approximation by suitable cylinder functions, $F = \lim_{n \to \infty} f_n$, i.e. via a time-lattice approximation

$$\int_{\mathcal{C}(Q,q_a)} \mathrm{d}M[x] \, F[x] = \lim_{n \to \infty} \int_{\mathcal{C}(Q,q_a)} \mathrm{d}M[x] \, f_n(x(t_n), \dots , x(t_1))$$

Properties:

- $\int_{\mathcal{C}(Q,q_a)} \mathrm{d}M[x] = 1$

- $\displaystyle\int_{\mathcal{C}(Q,q_a)} \mathrm{d}M[x]\,\delta(x(t_n)-q_n)\cdots\delta(x(t_1)-q_1) = m_{t_n-t_{n-1}}(q_n,q_{n-1})\cdots m_{t_1-t_0}(q_1,q_a)$

Generator T_M: Is local operator on dense subset of $L^2(Q)$ defined by its q'-representation

$$T_M^{(q')}\psi(q') = \langle q'|T_M|\psi\rangle \qquad \text{via} \qquad \langle q'|\mathrm{e}^{-tT_M}|q\rangle := m_t(q',q)$$

It obeys the

Fokker-Planck equation:

$$-\partial_t m_t(q',q) = T_M^{(q')} m_t(q',q)$$

Typically structure:

$$T_M = \frac{\vec{P}^{\,2}}{2} + \mathrm{i}\vec{P}\cdot\vec{m}(\vec{Q})$$

Diffusion constant $D = \frac{1}{2}$.
Drift $\vec{m}(\vec{Q})$ due to external force

Feynman-Kac formula: Let $H := T_M + V(Q)$ then

$$\boxed{\langle q_b|\mathrm{e}^{-tH}|q_a\rangle = \int_{\mathcal{C}(Q,q_a)} \mathrm{d}M[x]\,\delta(x(t)-q_b)\exp\left\{-\int_0^t \mathrm{d}\tau\,V(x(\tau))\right\}}$$

Conclusion: There are at least three equivalent ways (generator, transition density, path integral) to characterise a Markov process, they are related by

$$\langle q'|\mathrm{e}^{-tT_M}|q\rangle = m_t(q',q) = \int_{\mathcal{C}(Q,q)} \mathrm{d}M[x]\,\delta(x(t)-q')$$

Examples:
Wiener process
Bessel process
Legendre process