Lecture 5

21 Coherent states path integrals

21.1 Boson coherent states

Recall harmonic oscillator on $\mathcal{H} = L^2(\mathbb{R})$; $(m = \hbar = \omega = 1)$

$$H = \frac{P^2}{2} + \frac{Q^2}{2} = \frac{1}{2}(a^\dagger a + a a^\dagger)\,, \qquad a := \frac{1}{\sqrt{2}}(Q + \mathrm{i}P)\,, \qquad a^\dagger = \frac{1}{\sqrt{2}}(Q - \mathrm{i}P)$$

$$\implies \qquad [a, a^\dagger] = 1\,.$$

Number operator: $N = a^\dagger a$ with $N|n\rangle = n|n\rangle\,, \quad n = 0, 1, 2, 3, \ldots$

Ground state: $a|0\rangle = 0$

Excited states generated by $a^\dagger$: $|n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n|0\rangle$

Boson coherent state:

$$\boxed{|z\rangle := \mathrm{e}^{-\frac{1}{2}|z|^2}\,\mathrm{e}^{z a^\dagger}|0\rangle\,, \qquad z \in \mathbb{C}\,.}$$

Consider

$$a|z\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{n!}\,a\left(a^\dagger\right)^n|0\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{n!}\,n\left(a^\dagger\right)^{n-1}|0\rangle = z|z\rangle\,.$$

Coherent states are eigenstates of annihilation operator with complex eigenvalue z.
We may set $z = \frac{1}{\sqrt{2}}(q + \mathrm{i}p)$, which may be viewed as phase-space variable $(p, q) \in \mathbb{R}^2$.

Also note: $\langle n|z\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2}\dfrac{z^n}{\sqrt{n!}}$.

Over-Completeness:
Using BCH formula $\mathrm{e}^X\mathrm{e}^Y = \mathrm{e}^{[X,Y]}\mathrm{e}^Y\mathrm{e}^X$ if $[X, [X, Y]] = 0 = [Y, [X, Y]]$

$$\mathrm{e}^{\frac{1}{2}|z|^2 + \frac{1}{2}|z'|^2}\langle z|z'\rangle = \langle 0|\mathrm{e}^{z^* a}\mathrm{e}^{z' a^\dagger}|0\rangle = \langle 0|\mathrm{e}^{z^* z'[a,a^\dagger]}\mathrm{e}^{z' a^\dagger}\mathrm{e}^{z^* a}|0\rangle = \mathrm{e}^{z^* z}\langle 0|\mathrm{e}^{z a^\dagger}|0\rangle = \mathrm{e}^{z^* z'}$$

$$\boxed{\langle z|z'\rangle = \exp\left\{-\tfrac{1}{2}|z|^2 - \tfrac{1}{2}|z'|^2 + z^* z'\right\}}$$

Normalised: $\langle z|z\rangle = 1$

Resolution of unity:

$$\begin{aligned}
\int_{\mathbb{C}} \mathrm{d}^2 z\,\langle n|z\rangle\langle z|m\rangle &= \int_{\mathbb{C}} \mathrm{d}^2 z\,\mathrm{e}^{-|z|^2}\,\frac{z^n (z^*)^m}{\sqrt{n!m!}} \\
&= \int_0^\infty \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi\,\rho^{m+n+1}\mathrm{e}^{-\rho^2}\,\frac{\mathrm{e}^{\mathrm{i}(n-m)\varphi}}{\sqrt{n!m!}} \\
&= \frac{2\pi}{m!}\,\delta_{mn}\int_0^\infty \mathrm{d}t\,\frac{t^m}{2}\mathrm{e}^{-t} = \pi\,\delta_{mn}
\end{aligned}$$

$$\boxed{\int_{\mathbb{C}} \frac{\mathrm{d}^2 z}{\pi}\,|z\rangle\langle z| = 1}$$

General definition of arbitrary coherent states:

- Strong continuous in their label
- Over-completeness
- Resolution of unity

Time evolution:

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}Ht/\hbar}|z\rangle &= \sum_{n=0}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t(n+\frac{1}{2})}|n\rangle\langle n|z\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t(n+\frac{1}{2})}|n\rangle\, \mathrm{e}^{-\frac{1}{2}|z|^2}\frac{z^n}{\sqrt{n!}} \\
&= \mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}} \sum_{n=0}^{\infty} |n\rangle\, \mathrm{e}^{-\frac{1}{2}|z|^2}\frac{(z\mathrm{e}^{-\mathrm{i}\omega t})^n}{\sqrt{n!}} = \mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}} \sum_{n=0}^{\infty} |n\rangle\, \langle n|z\mathrm{e}^{-\mathrm{i}\omega t}\rangle = \mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}}|z\mathrm{e}^{-\mathrm{i}\omega t}\rangle
\end{aligned}$$

Label of coherent state follows classical eq. of motion: $z(t) = \mathrm{e}^{-\mathrm{i}\omega t}z(0)$, that is,

$$q(t) = q(0)\cos\omega t - p(0)\sin\omega t\,, \qquad p(t) = p(0)\cos\omega t + q(0)\sin\omega t\,.$$

Gaussian integral:

$$\int \frac{\mathrm{d}^2 z}{\pi}\,\mathrm{e}^{-a|z|^2} = 2\int_0^\infty \mathrm{d}r\, r\mathrm{e}^{-ar^2} = \int_0^\infty \mathrm{d}s\,\mathrm{e}^{-as} = \frac{1}{a}$$

For symmetric $\mathbf{A} = \mathbf{A}^\dagger > 0$, $\dim\mathbf{A} = n$

$$\int \frac{\mathrm{d}^2\vec{z}}{\pi^n}\,\mathrm{e}^{-\vec{z}^*\cdot\mathbf{A}\vec{z}} = \frac{1}{\det\mathbf{A}}$$

21.2 Boson coherent state path integral

Consider normal-ordered Hamiltonian $H(a^\dagger, a) = {:}H(a^\dagger, a){:}\,$.
Here, all a are to the right of all $a^\dagger$, then

$$\langle z|H(a^\dagger, a)|z'\rangle = H(z^*, z')\langle z|z'\rangle$$

Example: $\quad :aa^\dagger: = a^\dagger a$, $\qquad$ harm. osc. $H = :\frac{1}{2}(P^2+Q^2): = \hbar\omega a^\dagger a$

$$\mathrm{Tr}\,\mathrm{e}^{-\mathrm{i}Ht/\hbar} = \sum_{n=0}^{\infty}\mathrm{e}^{-\mathrm{i}\omega t n} = \frac{1}{1-\mathrm{e}^{-\mathrm{i}\omega t}} = \frac{\mathrm{e}^{\mathrm{i}\frac{\omega t}{2}}}{2\mathrm{i}\sin\frac{\omega t}{2}}$$

Consider Lie-Trotter

$$\mathrm{e}^{-\mathrm{i}Ht/\hbar} = \lim_{N\to\infty}\left(1 - \frac{\mathrm{i}}{\hbar}H\varepsilon\right)^N \qquad \text{with} \qquad t = N\varepsilon$$

insert $N-1$ times resolution of unity

$$\int \frac{\mathrm{d}^2 z_k}{\pi}\,|z_k\rangle\langle z_k| = 1$$

to arrive at

$$\langle z_N|\mathrm{e}^{-\mathrm{i}Ht/\hbar}|z_0\rangle = \lim_{N\to\infty}\prod_{k=1}^{N-1}\int\frac{\mathrm{d}^2 z_k}{\pi}\prod_{k=1}^{N}\underbrace{\langle z_k|z_{k-1}\rangle\left(1-\frac{\mathrm{i}}{\hbar}H(z_k^*, z_{k-1})\varepsilon\right)}_{=\mathrm{e}^{-\frac{1}{2}|z_k|^2}\mathrm{e}^{-\frac{1}{2}|z_{k-1}|^2}\mathrm{e}^{z_k^* z_{k-1}}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(z_k^*, z_{k-1})\varepsilon}}\,.$$

With

$$-\frac{1}{2}|z_k|^2 + \frac{1}{2}z_k^* z_{k-1} - \frac{1}{2}|z_{k-1}|^2 + \frac{1}{2}z_k^* z_{k-1} = -\frac{1}{2}z_k^*\Delta z_k + \frac{1}{2}\Delta z_k^* z_k$$

one finds

$$\begin{aligned}
\langle z''|\mathrm{e}^{-\mathrm{i}Ht/\hbar}|z'\rangle &= \lim_{N\to\infty}\prod_{k=1}^{N-1}\int\frac{\mathrm{d}^2 z_k}{\pi}\prod_{k=1}^{N}\exp\left\{\frac{1}{2}\Delta z_k^* z_k - \frac{1}{2}z_k^*\Delta z_k - \frac{\mathrm{i}}{\hbar}H(z_k^*, z_{k-1})\varepsilon\right\} \\
&= \int\mathcal{D}[z(\tau)]\,\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau\,\left[\frac{\hbar}{2\mathrm{i}}(\dot{z}^* z - z^*\dot{z}) - H(z^*, z)\right]\right\}
\end{aligned}$$

Imposing periodic boundary conditions (when looking on partition function), i.e. $z(0) = z(t)$ one may integrate by parts and action reads

$$S(t) = \int_0^t \mathrm{d}\tau \left[\frac{\hbar}{2\mathrm{i}}(\dot{z}^* z - z^* \dot{z}) - H(z^*, z)\right] = \int_0^t \mathrm{d}\tau \, [\mathrm{i}\hbar z^* \dot{z} - H(z^*, z)]$$

Consider harmonic oscillator $H(z^*, z) = \hbar\omega z^* z$ (normal ordering!)

$$\mathrm{Tr}\, \mathrm{e}^{-\mathrm{i}Ht/\hbar} = \int_{z(0)=z(t)} \mathcal{D}[z(\tau)] \, \exp\left\{\mathrm{i}\int_0^t \mathrm{d}\tau \, z^* \, (\mathrm{i}\partial_\tau - \omega)\, z\right\} = \frac{\mathcal{N}}{\det{(\mathrm{i}\partial_\tau - \omega)}}$$

Eigenvalues for periodic boundary: $\lambda_n = \frac{\pi}{t} 2n - \omega \,, \qquad n \in \mathbb{Z}\,.$ (Homework)

$$\frac{\mathcal{N}}{\det{(\mathrm{i}\partial_\tau - \omega)}} = \frac{\mathcal{N}'_B}{\frac{\omega t}{2}} \prod_{n\neq 0} \frac{1}{1 - \frac{\omega t}{2\pi n}} = \frac{\mathcal{N}'_B}{\frac{\omega t}{2}} \prod_{n=1}^{\infty} \frac{1}{1 - \left(\frac{\omega t}{2\pi n}\right)^2} = \frac{\mathcal{N}'_B}{\sin\frac{\omega t}{2}}$$

Obviously, $\mathcal{N}'_B = \mathrm{e}^{\mathrm{i}\frac{\omega t}{2}}/2\mathrm{i}$.

Comments:

- For anti-periodic boundary condition one would not get the correct result! (add. homework)
- RULE: For bosonic degrees of freedom always apply periodic boundary conditions

22 Path integrals for Fermions

22.1 Grassmann numbers

We have seen that complex numbers $z \in \mathbb{C}$ may in essence represent the classical phase space of a bosonic degree of freedom.
Similarly Grassmann numbers may represent the "classical phase space" of a fermionic degree of freedom.

Grassmann algebra: over the field of complex numbers is generated by the set $\{\overline{\psi}, \psi\}$.
$\overline{\psi}$ is "complex" conjugate of ψ and the algebra is define via anti-commutator

$$\{\psi, \overline{\psi}\} := \psi\overline{\psi} + \overline{\psi}\psi = 1\,, \qquad \{\psi, \psi\} = 0\,, \qquad \{\overline{\psi}, \overline{\psi}\} = 0\,.$$

That is, $\psi^2 = 0 = \overline{\psi}^2$.
Integration and differentiation are defined by

$$\int \mathrm{d}\psi \, 1 = 0\,, \qquad \int \mathrm{d}\psi \, \psi = 1\,, \qquad \frac{\mathrm{d}}{\mathrm{d}\psi}\, 1 = 0\,, \qquad \frac{\mathrm{d}}{\mathrm{d}\psi}\, \psi = 1\,.$$

ψ and $\overline{\psi}$ are odd Grassmann numbers (anti-commute) , while $\overline{\psi}\psi$ is an even (commuting) Grassmann number.

22.2 Fermion coherent state

Let $f, f^\dagger$ be fermion creation and annihilation operators, i.e.,

$$\{f, f^\dagger\} = 1\,, \qquad f^2 = 0 \qquad {f^\dagger}^2 = 0\,.$$

Fermion number operator $\mathcal{F} := f^\dagger f$ obeys

$$\mathcal{F}^2 = f^\dagger f f^\dagger f = f^\dagger(1 - f^\dagger f) f = f^\dagger f = \mathcal{F} \qquad \Longrightarrow \qquad \mathrm{spec}\, \mathcal{F} = \{0, 1\}\,.$$

Two-dimensional Fermion Hilbert space $\mathcal{H} = \mathbb{C}^2 = \mathrm{span}\{|0\rangle, |1\rangle\}$ with

$$\mathcal{F}|0\rangle = 0\,, \quad \mathcal{F}|1\rangle = |1\rangle \qquad \text{or} \qquad f|0\rangle = 0\,, \quad f^\dagger|0\rangle = 1\,, \quad f|1\rangle = |0\rangle\,, \quad f^\dagger|1\rangle = 0\,.$$

Define fermion coherent state in analogy to bosonic version:

$$\boxed{|\psi\rangle := \mathrm{e}^{-\frac{1}{2}\overline{\psi}\psi}\,\mathrm{e}^{f^\dagger\psi}\,|0\rangle = \mathrm{e}^{-\frac{1}{2}\overline{\psi}\psi}\,\mathrm{e}^{-\psi f^\dagger}\,|0\rangle = \mathrm{e}^{-\frac{1}{2}\overline{\psi}\psi}\,(|0\rangle - \psi|1\rangle)}$$

Note: ψ and f anti-commute, i.e., f and $f^\dagger$ are odd operators with respect to Grassmann algebra. Adjoint is given by

$$\langle\psi| := \mathrm{e}^{-\frac{1}{2}\overline{\psi}\psi}\,\langle 0|\mathrm{e}^{\overline{\psi}f} = \mathrm{e}^{-\frac{1}{2}\overline{\psi}\psi}\,\big(\langle 0| + \overline{\psi}\langle 1|\big)$$

Eigenstates: $\quad f|\psi\rangle = \psi|\psi\rangle \qquad \langle\psi|f^\dagger = \overline{\psi}\langle\psi|$

Proof: $f|\psi\rangle = \mathrm{e}^{-\frac{1}{2}\overline{\psi}\psi}\,\psi|0\rangle = \psi|\psi\rangle\,, \qquad \langle\psi|f^\dagger = \mathrm{e}^{-\frac{1}{2}\overline{\psi}\psi}\,\overline{\psi}\langle 0| = \overline{\psi}\langle\psi|$

Over-Completeness:

$$\boxed{\langle\psi|\psi'\rangle = \exp\left\{-\tfrac{1}{2}\overline{\psi}\psi - \tfrac{1}{2}\overline{\psi'}\psi'\right\}\big(1 + \overline{\psi}\psi\big) = \exp\left\{-\tfrac{1}{2}\overline{\psi}\psi - \tfrac{1}{2}\overline{\psi'}\psi' + \overline{\psi}\psi'\right\}}$$

Normalised: $\langle\psi|\psi\rangle = 1$.

Resolution of unity:

$$\mathrm{e}^{-\overline{\psi}\psi}\,(|0\rangle - \psi|1\rangle)\big(\langle 0| + \overline{\psi}\langle 1|\big) = \big(1 - \overline{\psi}\psi\big)\big(|0\rangle\langle 0| - \psi|0\rangle\langle 1| + \overline{\psi}|0\rangle\langle 1| - \overline{\psi}\psi|1\rangle\langle 1|\big)$$

$$\boxed{\int \mathrm{d}\overline{\psi}\mathrm{d}\psi\,|\psi\rangle\langle\psi| = 1}$$

Time evolution:
Let us consider fermionic oscillator Hamiltonian $H = \hbar\omega f^\dagger f = \hbar\omega\mathcal{F}$

$$\mathrm{e}^{-\mathrm{i}Ht/\hbar}|\psi\rangle = \mathrm{e}^{-\frac{1}{2}\overline{\psi}\psi}\mathrm{e}^{-\mathrm{i}\omega t\mathcal{F}}\,(|0\rangle - \psi|1\rangle) = \mathrm{e}^{-\frac{1}{2}\overline{\psi}\psi}\,\big(|0\rangle - \mathrm{e}^{-\mathrm{i}\omega t}\psi|1\rangle\big) = |\mathrm{e}^{-\mathrm{i}\omega t}\psi\rangle$$

Note $\overline{a\psi} = a^*\overline{\psi}$.

$$\mathrm{Tr}\,\mathrm{e}^{-\mathrm{i}Ht/\hbar} = 1 + \mathrm{e}^{-\mathrm{i}\omega t} = 2\mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}}\cos(\tfrac{\omega t}{2})\,.$$

Gaussian integral:

$$\int \mathrm{d}\overline{\psi}\mathrm{d}\psi\,\mathrm{e}^{-a\overline{\psi}\psi} = \int \mathrm{d}\overline{\psi}\mathrm{d}\psi\,(1 - a\overline{\psi}\psi) = a$$

For a symmetric matrix $\mathbf{A} = \mathbf{A}^\dagger$, $\dim\mathbf{A} = n$ (see, e.g.. Zinn-Justin)

$$\int \mathrm{d}\overline{\psi_1}\mathrm{d}\psi_1\cdots\mathrm{d}\overline{\psi_n}\mathrm{d}\psi_n\,\mathrm{e}^{-\vec{\overline{\psi}}\cdot\mathbf{A}\vec{\psi}} = \det\mathbf{A}$$

22.3 Fermion coherent state path integral

Again we consider a normal order Hamiltonian $H(f^\dagger, f) = :H(f^\dagger, f):$ for which we can write

$$\langle\psi|H(f^\dagger, f)|\psi'\rangle = H(\overline{\psi}, \psi')\langle\psi|\psi'\rangle$$

Consider Lie-Trotter

$$\mathrm{e}^{-\mathrm{i}Ht/\hbar} = \lim_{N\to\infty}\left(1 - \frac{\mathrm{i}}{\hbar}H\varepsilon\right)^N \qquad \text{with} \qquad t = N\varepsilon$$

insert $N-1$ times resolution of unity

$$\int \mathrm{d}\overline{\psi}_k \mathrm{d}\psi_k \, |\psi_k\rangle\langle\psi_k| = 1$$

to arrive at

$$\langle\psi_N|\mathrm{e}^{-\mathrm{i}Ht/\hbar}|\psi_0\rangle = \lim_{N\to\infty} \prod_{k=1}^{N-1} \int \mathrm{d}\overline{\psi}_k \mathrm{d}\psi_k \prod_{k=1}^{N} \underbrace{\langle\psi_k|\psi_{k-1}\rangle \left(1 - \frac{\mathrm{i}}{\hbar} H(\overline{\psi}_k, \psi_{k-1})\varepsilon\right)}_{=\mathrm{e}^{-\frac{1}{2}\overline{\psi}_k\psi_k}\mathrm{e}^{-\frac{1}{2}\overline{\psi}_{k-1}\psi_{k-1}}\mathrm{e}^{\overline{\psi}_k\psi_{k-1}}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(\overline{\psi}_k,\psi_{k-1})\varepsilon}} .$$

With

$$-\frac{1}{2}\overline{\psi}_k\psi_k - \frac{1}{2}\overline{\psi}_{k-1}\psi_{k-1} + \overline{\psi}_k\psi_{k-1} = -\frac{1}{2}\overline{\psi}_k\Delta\psi_k + \frac{1}{2}\overline{\Delta\psi}_k\psi_{k-1}$$

one finds

$$\begin{aligned}\langle\psi''|\mathrm{e}^{-\mathrm{i}Ht/\hbar}|\psi'\rangle &= \lim_{N\to\infty} \prod_{k=1}^{N-1} \int \mathrm{d}\overline{\psi}_k \mathrm{d}\psi_k \prod_{k=1}^{N} \exp\left\{\frac{1}{2}\overline{\Delta\psi}_k\psi_{k-1} - \frac{1}{2}\overline{\psi}_k\Delta\psi_k - \frac{\mathrm{i}}{\hbar}H(\overline{\psi}_k,\psi_{k-1})\varepsilon\right\} \\ &= \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \, \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau \, \left[\frac{\hbar}{2\mathrm{i}}(\dot{\overline{\psi}}\psi - \overline{\psi}\dot{\psi}) - H(\overline{\psi},\psi)\right]\right\}\end{aligned}$$

Here we impose anti-periodic boundary conditions (when looking on partition function), i.e. $\psi(0) = -\psi(t)$ and $\overline{\psi}(0) = -\overline{\psi}(t)$. Integration by parts results in action

$$S(t) = \int_0^t \mathrm{d}\tau \, \left[\frac{\hbar}{2\mathrm{i}}(\dot{\overline{\psi}}\psi - \overline{\psi}\dot{\psi}) - H(\overline{\psi},\psi)\right] = \int_0^t \mathrm{d}\tau \, \left[\mathrm{i}\hbar\overline{\psi}\dot{\psi} - H(\overline{\psi},\psi)\right]$$

Consider harmonic oscillator $H = \hbar\omega f^\dagger f$, that is $H(\overline{\psi},\psi) = \hbar\omega\overline{\psi}\psi$

$$\mathrm{Tr}\,\mathrm{e}^{-\mathrm{i}Ht/\hbar} = \int_{\psi(0)=-\psi(t)} \mathcal{D}\overline{\psi}\mathcal{D}\psi \, \exp\left\{\mathrm{i}\int_0^t \mathrm{d}\tau\,\overline{\psi}\,(\mathrm{i}\partial_\tau - \omega)\,\psi\right\} = \mathcal{N}\,\det(\mathrm{i}\partial_\tau - \omega)$$

Eigenvalues for anti-periodic boundary: $\lambda_n = \frac{\pi}{t}(2n+1) - \omega\,, \qquad n \in \mathbb{Z}\,.$

$$\mathcal{N}\det(\mathrm{i}\partial_\tau - \omega) = \mathcal{N}'\prod_{n\in\mathbb{Z}}\left(1 - \frac{\omega t}{\pi(2n+1)}\right) = \mathcal{N}'\prod_{n=1}^{\infty}\left(1 - \left(\frac{\omega t}{\pi(2n+1)}\right)^2\right) = \mathcal{N}'\cos\frac{\omega t}{2}$$

Obviously $\mathcal{N}' = 2\mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}}$.

Comments:

- For periodic boundary condition one would not get the correct result! (see boson case)
- RULE: For fermionic degrees of freedom always apply anti-periodic boundary conditions when calculating the trace (partition function)

22.4 Witten index

Again, let $H = \hbar\omega f^\dagger f = \hbar\omega\mathcal{F}$ and consider

$$\mathrm{Tr}\,\left((-1)^{\mathcal{F}}\mathrm{e}^{-\mathrm{i}Ht/\hbar}\right) = 1 - \mathrm{e}^{-\mathrm{i}\omega t} = 2\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}}\sin\frac{\omega t}{2}$$

Obviously, this quantity (called Witten index) is represented by a path integral with periodic boundary conditions

$$\mathrm{Tr}\,\left((-1)^{\mathcal{F}}\mathrm{e}^{-\mathrm{i}Ht/\hbar}\right) = \int_{\psi(0)=\psi(t)} \mathcal{D}\overline{\psi}\mathcal{D}\psi \, \exp\left\{\mathrm{i}\int_0^t \mathrm{d}\tau\,\overline{\psi}\,(\mathrm{i}\partial_\tau - \omega)\,\psi\right\} = \mathcal{N}_F'\sin\frac{\omega t}{2}$$

where $\mathcal{N}'_F = 2\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}}$.
Observation: $\mathcal{N}'_B \frac{1}{\sin\frac{\omega t}{2}}\, \mathcal{N}'_F \sin\frac{\omega t}{2} = 1$!!! $\Longrightarrow$
Consider supersymmetric Hamiltonian $H_{SUSY} := \hbar\omega\left(a^\dagger a + f^\dagger f\right)$.
Witten index:

$$\Delta := \mathrm{Tr}\left((-1)^{\mathcal{F}}\mathrm{e}^{-\mathrm{i}tH_{SUSY}/\hbar}\right) = \mathrm{Tr}\left((-1)^{f^\dagger f}\mathrm{e}^{-\mathrm{i}\omega t f^\dagger f}\right)\mathrm{Tr}\left(\mathrm{e}^{-\mathrm{i}\omega t a^\dagger a}\right) = 1$$

For supersymmetric quantum systems the Witten index is a constant (topological index) and thus independent of any parameters (here ω) of the Hamiltonian.
The Witten index is represented by a path integral with periodic boundary conditions

$$\Delta = \int_{z(0)=z(t)} \mathcal{D}[z(\tau)] \int_{\psi(0)=\psi(t)} \mathcal{D}\overline{\psi}\mathcal{D}\psi\, \exp\left\{\mathrm{i}\int_0^t \mathrm{d}\tau\, \mathrm{i}(z^*\dot{z} + \overline{\psi}\dot{\psi}) - \omega(z^*z + \overline{\psi}\psi)\right\}.$$