Path Integrals

Lecture Notes

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Preliminaries

Dates:
Six Mondays 15.04.24, 22.04.24, 29.04.24, 06.05.24, 13.05.24, 27.05.24 (Test?)
Lecture 9 - 12, Tutorial 13 - 15, Homework Problems
Script and other details are available at
`https://www.eso.org/~gjunker/VorlesungSS2024.html`
Contact: `gjunker@eso.org`

Literature:

- Feynman R.P. and Hibbs A.R. 2010 *Quantum Mechanics and Path Integrals* (Mineola: Dover Publication) emended by D.F. Styer
- Schulman 2005 *Techniques and Applications of Path Integration* (Mineola: Dover Publication)
- Kleinert H. 2009 *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* (Singapore: World Scientific)
- Roepstorff G. 1994 *Path Integral Approach to Quantum Physics* (Berlin: Springer)
- Zinn-Justin J. 2005 *Path Integrals in Quantum Mechanics* (Oxford: University Press)
- Moshayedi N. 2023, *Quantum Field Theory and Functional Integrals* (Berlin: Springer)
- Felsager B. 1998, *Geometry, Particles and Fields* (Berlin: Springer)

Path integrals are sometimes called the third way towards quantum mechanics. Based on first ideas of Dirac, Feynman developed the idea of "path integrals" as an alternative way towards quantum mechanics in his Ph.D. thesis.

Dirac (1933), *The Lagrangian in quantum mechanics*, Physik. Zeits. Sowjetunion, 3, 64–72.
Dirac (1945), *On the analogy between classical and quantum mechanics*, RMP 17, 195–199.
Feynman (1948), *Space-time approach to non-relativistic quantum mechanics*, RMP 20, 367–387.

and the forgotten work by Wentzel much earlier

Wentzel (1924), *Zur Quantenoptik*, Zeits. für Physik 22, 193–199.
Wentzel (1924), *Zur Quantentheorie des Röntgenbremsspektrums*, Zeits. für Physik 27, 257–284.

Lecture 1

1 Concepts of classical mechanics

Let's consider a non-relativistic point-like particle of mass $m > 0$ moving along the real line $\mathbb{R}$ under the influence of an external force characterised by potential V.

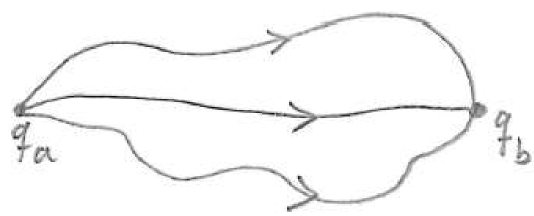
• Lagrangian:

$$L(\dot{q}, q) := \frac{m}{2}\,\dot{q}^2 - V(q)\,, \qquad q \in \mathbb{R}$$

• Action:

$$S[x(t)] := \int_{t_a}^{t_b} \mathrm{d}t\, L(\dot{x}(t), x(t))$$

Is a *functional* of paths which start at $q_a := x(t_a)$ and end at $q_b := x(t_b)$.



• Hamilton's principle:
Classical dynamics follows paths $x(t)$ along which the action is stationary

$$\frac{\delta S[x(t)]}{\delta x(t)} = 0$$

• Euler-Lagrange equation: Follows from Hamilton's principle

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

Characterises the dynamics in configuration space $q \in \mathbb{R}$

• Hamilton function: The energy of the system expressed in terms of coordinates and momenta

$$H(p, q) := p\dot{q} - L(\dot{q}, q)$$

with canonical momentum $p := \frac{\partial L}{\partial \dot{q}}$ and $\dot{q} = \dot{q}(p, q)$ [Legendre transformation]

• Hamilton's equations:

$$\dot{q} = \frac{\partial H(p, q)}{\partial p}\,, \qquad \dot{p} = -\frac{\partial H(p, q)}{\partial q}\,.$$

Characterises the dynamics in phase space $(p, q) \in \mathbb{R}^2$.

• Poisson brackets:
Let $F(p, q)$ and $G(p, q)$ be phase-space functions then

$$\{F, G\}_{\mathrm{PB}} := \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial G}{\partial q}\frac{\partial F}{\partial p} = -\{G, F\}_{\mathrm{PB}}$$

In particular we have $\{q, p\}_{\mathrm{PB}} = 1$.

• General dynamics for phase-space functions:

$$\dot{F}(p, q, t) = \{F, H\}_{\mathrm{PB}} + \frac{\partial F}{\partial t}$$

Hamilton function H acts as generator of the dynamics (time evolution)

2 Canonical quantisation

Heisenberg (1925): Matrizenmechnik (Heisenberg picture)
Schrödinger (1926): Wellenmechanik (Schrödinger picture)

Canonical quantisation:

Phase space $\mathbb{R}^2$	$\longrightarrow$	Hilbert space $\mathcal{H} = L^2(\mathbb{R})$
Phase space functions F	$\longrightarrow$	Operators acting on $\mathcal{H}$, $\hat{F} : \mathcal{H} \to \mathcal{H}$
Poisson bracket $\{F, G\}_{\text{PB}}$	$\longrightarrow$	Commutator $\frac{1}{\mathrm{i}\hbar}[\hat{F}, \hat{G}]$, e.g. $[\hat{Q}, \hat{P}] = \mathrm{i}\hbar$

Quantum dynamics: Heisenberg picture

$$\dot{\hat{F}} = \frac{1}{\mathrm{i}\hbar}\left[\hat{F}, \hat{H}\right] + \frac{\partial \hat{F}}{\partial t}$$

When $\hat{F}$ is not explicitly depending on time t then this integrates to

$$\hat{F}(t) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}t}\hat{F}(0)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}$$

Proof: $\frac{\mathrm{d}}{\mathrm{d}t}\hat{F} = \frac{\mathrm{i}}{\hbar}\hat{H}\hat{F} - \frac{\mathrm{i}}{\hbar}\hat{F}\hat{H} = \frac{\mathrm{i}}{\hbar}[\hat{F}, \hat{H}]$

Expectation values: For system in pure state $|\psi\rangle \in \mathcal{H}$

$$\langle \hat{F} \rangle_\psi := \langle \psi | \hat{F}(t) | \psi \rangle = \langle \psi | \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}t}\hat{F}(0)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t} | \psi \rangle = \langle \psi_S(t) | \hat{F}(0) | \psi_S(t) \rangle$$

with

$$|\psi_S(t)\rangle := \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}|\psi\rangle \qquad \text{Schrödinger picture}$$

Schrödinger equation: Schrödinger picture

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi_S(t)\rangle = \hat{H}\,|\psi_S(t)\rangle = H(\hat{P}, \hat{Q})\,|\psi_S(t)\rangle$$

Ordering problems may occur in $H(\hat{P}, \hat{Q})$

q-representation: $\psi_S(q, t) :=\rangle \langle q | \psi_S(t) \rangle$ where $\langle q | \hat{Q} = q \langle q |$, $\quad \langle q | \hat{P} = \left(-\mathrm{i}\hbar\frac{\partial}{\partial q}\right)\langle q |$ $\qquad \Longrightarrow$

$$i\hbar\frac{\partial}{\partial t}\psi_S(q, t) = H\left(-\mathrm{i}\hbar\frac{\partial}{\partial q}, q\right)\psi_S(q, t)$$

For standard systems $H(p, q) = \frac{p^2}{2m} + V(q)$ the quantum Hamiltonian in q-repreesentation reads

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)$$

Integration of Schrödinger equation results in

$$\psi_S(q, t) = \langle q | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t} | \psi \rangle$$

Insert complete set of $\hat{Q}$-eigenstates

$$\int_{-\infty}^{+\infty} \mathrm{d}q_0\,|q_0\rangle\langle q_0| = \hat{1}$$

leads us to

$$\psi_S(q, t) = \int_{-\infty}^{+\infty} \mathrm{d}q_0\,\underbrace{\langle q | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t} | q_0 \rangle}_{\text{Integral kernel}}\,\psi_S(q_0, 0)$$

The integral kernel represents the transition amplitude for the quantum particle starting in q_0 at $t = 0$ to arrive at q after time $t > 0$. It is also called the quantum **propagator**.

3 The propagator

Is the key object of Feynman's path integral

Definition:

$$K(x'', x'; t) := \langle x''|\hat{U}(t)|x'\rangle = \langle x''|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}|x'\rangle$$

q-representation of the time evolution $\hat{U}(t) := \exp\left\{-\frac{\mathrm{i}}{\hbar}\hat{H}t\right\}$

Properties:

- Unitarity: $\hat{U}^\dagger(t)\hat{U}(t) = \hat{1} = \hat{U}(t)\hat{U}^\dagger(t)$
- Convolution: $\hat{U}(t-\tau)\hat{U}(\tau - t_0) = \hat{U}(t - t_0)$

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, K(x'', x; t-\tau)K(x, x'; \tau - t_0) = K(x'', x'; t - t_0)$$

- Spectral representation: Assume $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$, pure point spectrum

$$\hat{U}(t) = \sum_n \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n t}|\psi_n\rangle\langle\psi_n|$$

$$K(x'', x'; t) = \sum_n \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n t}\psi_n(x'')\psi_n^*(x')$$

- Green function

$$\hat{G}(E) := \frac{1}{\hat{H} - E}\,, \qquad E \in \mathbb{C}\backslash \mathrm{spec}\,(\hat{H})$$

$$\begin{aligned} G(x'', x'; E) &:= \langle x''|\hat{G}(E)|x'\rangle \\ &= \int_0^\infty \mathrm{d}t\, \frac{\mathrm{i}}{\hbar}\langle x''|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\hat{H}-E)t}|x'\rangle\,, \qquad \mathrm{Im}\, E > 0 \\ &= \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}Et}K(x'', x'; t) \\ &= \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}t\, P_E(x'', x'; t) \end{aligned}$$

with promotor

$$P_E(x'', x'; t) := \langle x''|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\hat{H}-E)t}|x'\rangle$$

Will be relevant at later stage when changing space and time variables for explicit solutions.

4 Some historical background

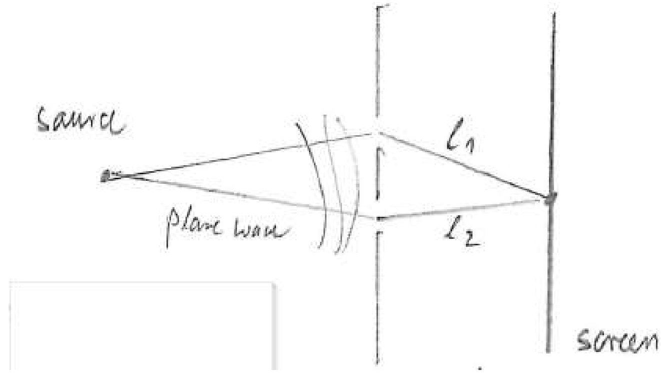
N. Wiener 1923: First "path integral" to describe statistical properties of Brownian motion.

G. Wenzel 1924: In connection with quantum optics introduces a "sum of paths" weighted by phase factors.

P.A.M. Dirac 1933: His basic argument is as follows. Canonical quantisation is based on Hamilton's theory of classical mechanics. The alternative based on the Lagrangian is (believed to be) more fundamental.
So he comes up with the question "What role plays the Lagrangian in quantum mechanics? His basic idea is that the transition amplitude

$$\langle\psi_S(t)|\psi_S(0)\rangle \qquad \text{"corresponds to"} \qquad \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau L(\dot{q}, q)\right\}$$

Recall double slit experiment:



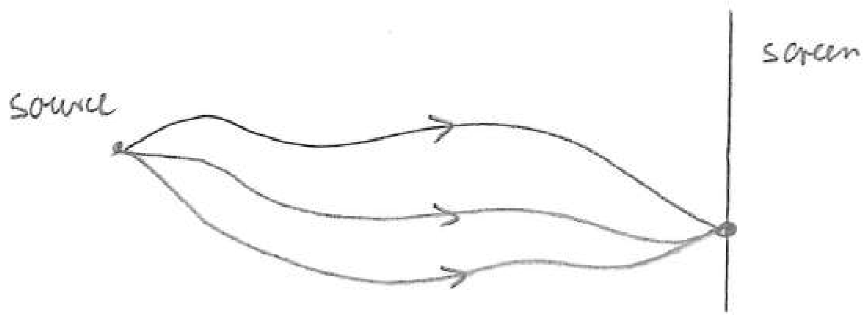
Photons: Wave amplitude arriving at a given position on screen (interference) is given by

$$\sim \mathrm{e}^{\mathrm{i}k\ell_1} + \mathrm{e}^{\mathrm{i}k\ell_2}\,, \qquad k = 2\pi/\lambda\,, \qquad \text{wave number}$$

Electrons: Show same interference pattern, so wave amplitude arriving at a given position on screen is also a sum of phases

$$\sim \Phi_1 + \Phi_2\,, \qquad \text{passage thru slit 1 and 2, respectively}$$

Taking the double slit away = slits everywhere



Interpretation/assumption: Electron (wave) takes all paths simultaneously. Each path contributes with same amplitude and a path-dependent phase $\Phi[x(t)]$ to the transition amplitude

$$K(x'', x'; t) = \sum_{\substack{\text{paths } x(\tau) \\ x'=x(0) \\ x''=x(\tau)}} \Phi[x(\tau)]$$

R. Feynman 1948: In essence, replaced Dirac's "corresponds to" by an equal sign.

"All paths contribute with same amplitude and a phase proportional to the action"

$$\Phi[x(\tau)] = \text{const.} \exp\left\{\frac{\mathrm{i}}{\hbar} S[x(\tau)]\right\}$$

Big Problem: How this sum over paths shall be interpreted or even be performed!?

$\Longrightarrow$ Path Integral, Functional Integral, Sum over Histories

5 Derivation of Feynman's path integral

Consider propagator: (No more "hats" on operators)

$$K(x'', x'; t) = \langle x''|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht}|x'\rangle$$

Divide time interval into N slices $\varepsilon := T/N$

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\varepsilon}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\varepsilon}\cdots\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\varepsilon}$$

Insert $N-1$-times resolution of unity

$$\int_{-\infty}^{+\infty} \mathrm{d}x_j\,|x_j\rangle\langle x_j| = 1\,, \qquad j = 1, 2, \ldots, N-1$$

and take limit $N \to \infty$

$$K(x'', x'; t) = \lim_{N\to\infty} \int \mathrm{d}x_{N-1}\cdots\int \mathrm{d}x_1 \prod_{j=1}^{N} K(x_j, x_{j-1}; \varepsilon)\,,$$

with $x' = x_0$, $x'' = x_N$ and short-time propagator

$$K(x_j, x_{j-1}; \varepsilon) = \langle x_j|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\varepsilon}|x_{j-1}\rangle\,, \qquad \text{with} \qquad H = T + V = \frac{P^2}{2m} + V(Q)$$

Lie-Trotter formula: Exercise 1

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\varepsilon} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}T\varepsilon}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V\varepsilon}\left(1 + O(\varepsilon^2)\right)$$

Proof: $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\varepsilon} = 1-\frac{\mathrm{i}}{\hbar}(T+V)\varepsilon+O(\varepsilon^2) = (1-\frac{\mathrm{i}}{\hbar}T\varepsilon)(1-\frac{\mathrm{i}}{\hbar}V\varepsilon)(1+O(\varepsilon^2)) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}T\varepsilon}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V\varepsilon}\left(1 + O(\varepsilon^2)\right)$.
See also Baker–Campbell–Hausdorff or Zassenhaus formula: $\mathrm{e}^{\delta(T+V)} = \mathrm{e}^{\delta T}\mathrm{e}^{\delta V}(1 + O(\delta^2))$

On short time scale $\varepsilon \to 0$ we can commute T and V with error of $O(\varepsilon^2) \Longrightarrow$ Classical behaviour

Short time propagator:

$$K(x_j, x_{j-1}; \varepsilon) \simeq \langle x_j|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}T\varepsilon}|x_{j-1}\rangle\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V(x_{j-1})\varepsilon} \simeq \langle x_j|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}T\varepsilon}|x_{j-1}\rangle\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V(x_j)\varepsilon}$$

Free Particle Propagator: Exercise 2

$$\langle x''|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{P^2}{2m}t}|x'\rangle = \sqrt{\frac{m}{2\pi\mathrm{i}\hbar t}}\exp\left\{\frac{\mathrm{i}}{\hbar}\frac{m(x''-x')^2)}{2t}\right\}$$

Hence

$$\langle x_j|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}T\varepsilon}|x_{j-1}\rangle = \sqrt{\frac{m}{2\pi\mathrm{i}\hbar\varepsilon}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{m(\Delta x_j)^2}{2\varepsilon}}\,, \qquad \Delta x_j := x_j - x_{j-1}$$

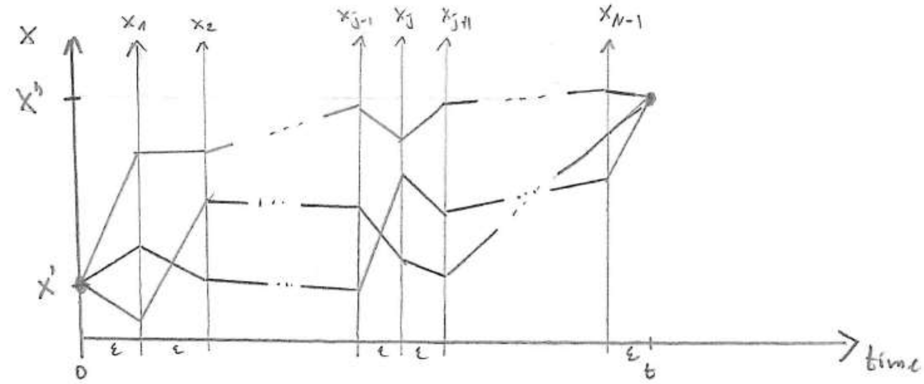
Result:

$$\boxed{K(x'', x'; t) = \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\hbar\varepsilon}\right)^{N/2}\int \mathrm{d}x_{N-1}\cdots\int \mathrm{d}x_1 \exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\varepsilon\left[\frac{m}{2}\left(\frac{\Delta x_j}{\varepsilon}\right)^2 - V(x_{j-1})\right]\right\}}$$

Note the appearance of the short-time action $S_j := \frac{m}{2}\frac{(\Delta x_j)^2}{\varepsilon} - V(x_{j-1})\varepsilon$
Formal expressions which are often used

$$K(x'', x'; t) = \int_{x(0)=x'}^{x(t)=x''} \mathcal{D}x(\tau)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[x(\tau)]} = \int_{x(0)=x'}^{x(t)=x''} \mathcal{D}x(\tau)\,\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau L(\dot{x}, x)\right\}$$

Interpretation:
In between kicks by potential $\mathrm{e}^{-(\mathrm{i}/\hbar)V(x_j)\varepsilon}$ the system moves freely for a very short time ε.



Severe problems:

- $\mathcal{D}x(\tau)$ is NOT a measure !!! $\Longrightarrow$ Wiener measure.
  Only the discrete time formalism is well-defined but..
- The limit $N \to \infty$ may not exist in general.
- Feynman could solve only the free particle and harmonic oscillator problem.
- A priori in cartesian coordinates.
- The Coulomb problem (once being the most important success of quantum mechanic) could not be solved for a long time
- H-atom formally solved in 1979/1982

Early successes:

- In QFT and in particular QED
  (Nobel Prize in Physics 1965 to Sin-Itiro Tomonaga, Julian Schwinger, Richard P. Feynman)
- Perturbation theory (Feynman diagrams)
- Elimination of oscillators
- Renormalization group

6 Rules for explicit calculations

- Regularisation of oscillatory integrals, for example, via $m \to m + \mathrm{i}\eta$ and let $\eta \searrow 0$ in the end

  $$\exp\left\{\frac{\mathrm{i}m\Delta x_j^2}{2\hbar\varepsilon}\right\} \qquad \text{can be integrated}$$

  Feynman original proposed $\hbar \to \hbar(1 - \mathrm{i}\delta)$ with $\delta \searrow 0$.

  Alternative is a Wick rotation $t \to -\mathrm{i}\beta\hbar$

  $$\mathrm{e}^{-(\mathrm{i}/\hbar)Ht} \to \mathrm{e}^{-\beta H}$$

  Leads to the well-defined Wiener measure in limit $N \to \infty$ (stochastic processes).

- All terms of order $O(\varepsilon^{1+\delta})$, $\delta > 0$, may be ignored in short-time action
- $\Delta x_j = O(\varepsilon^{1/2})$ due to the "Gaussian" weight $\sqrt{\frac{m}{2\pi i\hbar\varepsilon}} \exp\left\{\frac{im\Delta x_j^2}{2\hbar\varepsilon}\right\}$. See Homework

$$\frac{1}{\sqrt{\pi\varepsilon}} \int dx\, x^{2n} e^{-x^2/\varepsilon} = \left(\frac{\varepsilon}{2}\right)^n (2n-1)!!$$

  from which one can deduce

$$\int dx\, e^{-\frac{a}{\varepsilon}x^2 + \frac{b}{\varepsilon}x^4} = \int dx\, e^{-\frac{a}{\varepsilon}x^2 + \frac{3b}{4a^2}\varepsilon + O(\varepsilon^2)}$$

  Therefore we can approximate

$$S_j = \frac{m}{2}\frac{\Delta x_j^2}{\varepsilon} + \beta\frac{\Delta x_j^4}{\varepsilon} \approx \frac{m}{2}\frac{\Delta x_j^2}{\varepsilon} - \beta\frac{3\hbar^2}{4m^2}\varepsilon$$

  Here $a = -\frac{im}{2\hbar}$ and $b = \frac{i\beta}{\hbar}$. This is needed when changing variables (e.g. polar coordinates)

- Vector potentials: $\vec{B} = \vec{\nabla} \times \vec{A}$

$$L(\dot{\vec{x}}, \vec{x}) = \frac{m}{2}\dot{\vec{x}}^2 + \frac{e}{c}\dot{\vec{x}} \cdot \vec{A} - V(\vec{x})$$

  In short-time notation $\int dt\, \dot{\vec{x}} \cdot \vec{A} \to \Delta\vec{x}_j \cdot \vec{A}(\tilde{\vec{x}}_j)$. Hence choice of mid-point $\tilde{\vec{x}}_j$ becomes relevant as $\Delta\vec{x}_j = O(\sqrt{\varepsilon})$.

- Scalar potentials:

$$\begin{aligned} V(x_j)\varepsilon &= V(x_{j-1} + \Delta x_j)\varepsilon = V(x_{j-1})\varepsilon + V'(x_j)\underbrace{\Delta x_j \varepsilon}_{O(\varepsilon^{3/2})} + O(\varepsilon^2) \\ &\approx V(x_{j-1})\varepsilon \approx V(\overline{x}_j)\varepsilon \approx V(\hat{x}_j)\varepsilon \end{aligned}$$

  with $\overline{x}_j := \frac{1}{2}(x_j + x_{j-1})$ and $\hat{x}_j := \sqrt{x_j x_{j-1}}$ etc.

7 The free particle propagator

In tutorial we show (w/o path integrals)

$$\boxed{K_0(x'', x'; t) := \langle x''| e^{-\frac{i}{\hbar}\frac{P^2}{2m}t} |x'\rangle = \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left\{\frac{i}{\hbar}\frac{m}{2}\frac{(x''-x')^2}{t}\right\}}$$

Calculating the path integral

$$K_0(x'', x'; t) = \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{N/2} \int dx_{N-1} \cdots \int dx_1 \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\frac{m}{2}\frac{\Delta x_j^2}{\varepsilon}\right\}$$

Obviously

$$K_0(x'', x'; t) = \lim_{N\to\infty} \int dx_{N-1} \cdots \int dx_1 K_0(x_N, x_{N-1}; \varepsilon) \cdots K_0(x_1, x_0; \varepsilon)$$

Integration and limit trivial with the help of homework problem 5 resulting in above finite time expression.
In D dimension this reads

$$K_0(\vec{x}'', \vec{x}'; t) = \left(\frac{m}{2\pi i\hbar t}\right)^{D/2} \exp\left\{\frac{i}{\hbar}\frac{m}{2t}(\vec{x}'' - \vec{x}')^2\right\}.$$

Consider the classical action of free particle:
With

$$\dot{x}_{\rm cl} = \frac{x'' - x'}{t} = \text{const.}$$

we have

$$S_{\rm cl}(x'', x'; t) := S[x_{\rm cl}(\tau)] = \int_0^t \mathrm{d}\tau \, \frac{m}{2} \, \dot{x}_{\rm cl}(\tau) = \frac{m}{2} \frac{(x'' - x')^2}{t} \, .$$

Consider the expression

$$\frac{\partial^2 S_{\rm cl}(x'', x'; t)}{\partial x'' \partial x'} = -\frac{m}{t} \, .$$

This allows us to write

$$K_0(x'', x'; t) = \sqrt{\frac{\mathrm{i}}{2\pi\hbar} \frac{\partial^2 S_{\rm cl}(x'', x'; t)}{\partial x'' \partial x'}} \, \exp\left\{ \frac{\mathrm{i}}{\hbar} S_{\rm cl}(x'', x'; t) \right\} \, .$$

In D dimension it is given by

$$K_0(\vec{x}'', \vec{x}'; t) = \left(\frac{\mathrm{i}}{2\pi\hbar} \right)^{D/2} \sqrt{\det \left(\frac{\partial^2 S_{\rm cl}(\vec{x}'', \vec{x}'; t)}{\partial x''_a \partial x'_b} \right)} \, \exp\left\{ \frac{\mathrm{i}}{\hbar} S_{\rm cl}(\vec{x}'', \vec{x}'; t) \right\} \, .$$

with $\vec{x}^T = (x_1, x_2, \ldots, x_a, \ldots, x_D)$.

This is known under the name of **VanVleck-Pauli-Morette formula**.
Is exact for quadratic systems.
Is quasi-classical approximation for general systems.

8 Phase-space path integrals

Recall

$$K(x'', x'; t) = \lim_{N \to \infty} \int \mathrm{d}x_{N-1} \cdots \int \mathrm{d}x_1 \prod_{j=1}^{N} K(x_j, x_{j-1}; \varepsilon) \, ,$$

with short-time propagator

$$K(x_j, x_{j-1}; \varepsilon) = \langle x_j | \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} \frac{P^2}{2m} \varepsilon} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} V(Q) \varepsilon} | x_{j-1} \rangle \, .$$

Inserting unity in form of momentum eigenstates: $P|p_j\rangle = p_j|p_j\rangle$

$$1 = \int_{-\infty}^{\infty} \mathrm{d}p_j \, |p_j\rangle\langle p_j|$$

leads us to

$$\begin{aligned} K(x_j, x_{j-1}; \varepsilon) &= \int_{-\infty}^{\infty} \mathrm{d}p_j \, \langle x_j | p_j \rangle \langle p_j | x_{j-1} \rangle \, \exp\left\{ -\frac{\mathrm{i}}{\hbar} \left(\frac{p_j^2}{2m} + V(x_{j-1}) \right) \varepsilon \right\} \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \mathrm{d}p_j \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} x_j p_j} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} x_{j-1} p_j} \exp\left\{ -\frac{\mathrm{i}}{\hbar} \left(\frac{p_j^2}{2m} + V(x_{j-1}) \right) \varepsilon \right\} \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \mathrm{d}p_j \, \exp\left\{ \frac{\mathrm{i}}{\hbar} \left(p_j \frac{\Delta x_j}{\varepsilon} - \frac{p_j^2}{2m} - V(x_{j-1}) \right) \varepsilon \right\} \end{aligned}$$

Phase-space path integral:

$$K(x'', x'; t) = \lim_{N \to \infty} \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} \mathrm{d}x_j \prod_{j=1}^{N} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_j}{2\pi\hbar} \prod_{j=1}^{N} \exp\left\{ \frac{\mathrm{i}}{\hbar} \left(p_j \frac{\Delta x_j}{\varepsilon} - \frac{p_j^2}{2m} - V(x_{j-1}) \right) \varepsilon \right\}$$

Formal expression

$$K(x'', x'; t) = \int \mathcal{D}x\mathcal{D}p \, \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_0^t \mathrm{d}\tau \, (p\dot{x} - H(p,x)) \right\}$$

Remarks:

- In phase space paths are not continuous and thus this expression has even more severe problems.
- There is no well-defined mathematical counterpart available.
- Coherent states path integral should be used instead.
  Coherent state: $a|z\rangle = z|z\rangle \, , \; z \in \mathbb{C} \, , \qquad a := \frac{1}{\sqrt{2}}(Q + \mathrm{i}P) \, , \qquad z = (q + \mathrm{i}p)\sqrt{2} \, .$