Problem 17:

$$W_t(q_2,q_1) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(q_2-q_1)^2}{2t}}$$

a) $W_t(q_2,q_1) \geq 0$ obvious #

b) $\int_{\mathbb{R}} dq_2\, W_t(q_2,q_1) = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} = 1$ #

c) $\lim_{\varepsilon \searrow 0} \int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{2\pi\varepsilon^2}} e^{-\frac{x^2}{2\varepsilon^2}} \underset{\text{smooth function}}{f(x)} \underset{y=x/\varepsilon}{=} \lim_{\varepsilon\searrow 0} \int dy \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} f(y\varepsilon)$

$= \int_{-\infty}^{+\infty} dy \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \lim_{\varepsilon\searrow 0} f(y\varepsilon) = f(0)$ ⇝ approx. of δ-function

d) same calculation as in problem 5 ! 6

e) $\frac{1}{t}\int_{|x|>\varepsilon} dx \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} = \frac{2}{t}\int_{\varepsilon}^{\infty} dx \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \underset{y=\frac{x}{\sqrt{2t}}}{=} \frac{1}{t} \operatorname{erfc}\left(\frac{\varepsilon}{\sqrt{2t}}\right)$

$= \frac{1}{t} \frac{e^{-\frac{\varepsilon^2}{2t}}}{\varepsilon/\sqrt{2t}\,\sqrt{\pi}} \left(1 + O\left(\frac{2t}{\varepsilon^2}\right)\right) \approx \sqrt{\frac{2}{\varepsilon\pi t}}\, e^{-\frac{\varepsilon^2}{2t}} \left(1 + O(t)\right) \underset{t\searrow 0}{\longrightarrow} 0$

Problem 18:

$$W_t^{\pm}(q_2,q_1) = W_t^{(1)}(q_2,q_1) \pm W_t^{(1)}(-q_2,q_1) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{q_2^2+q_1^2}{2t}} \left(e^{\frac{q_2 q_1}{t}} \pm e^{-\frac{q_2 q_1}{t}}\right)$$

Hint

$$= \frac{1}{t}\sqrt{q_1 q_2}\, e^{-\frac{q_2^2+q_1^2}{2t}} I_{\mp\frac{1}{2}}\left(\frac{q_2 q_1}{t}\right)$$

from b)

$$b_t^{(-\frac{1}{2})}(q_2,q_1) = \frac{\sqrt{q_1 q_2}}{t} e^{-\frac{q_2^2+q_1^2}{2t}} I_{-\frac{1}{2}}\left(\frac{q_2 q_1}{t}\right) = W_t^{+}(q_2,q_1)$$

$$b_t^{(+\frac{1}{2})}(q_2,q_1) = \frac{q_2}{t}\sqrt{\frac{q_2}{q_1}}\, e^{-\frac{q_2^2+q_1^2}{2t}} I_{+\frac{1}{2}}\left(\frac{q_2 q_1}{t}\right) = \frac{q_2}{q_1} W_t^{-}(q_2,q_1)$$

$$\Rightarrow \quad w_t^+(q_2,q_1) = b_t^{(-\frac{1}{2})}(q_2,q_1)$$

therefore w_t^+ fullfils all condition in 17 as $b_t^{(\nu)}$ does

$$\Rightarrow \quad w_t^-(q_2,q_1) = \frac{q_1}{q_2}\, b_t^{(+\frac{1}{2})}(q_2,q_1)$$ fullfils all but not the normalisation

w_t^+ is also called the <u>reflecting</u> Wiener density on $\mathbb{R}^+$

as $$w_t^+(0,q_1) = \frac{2}{\sqrt{2\pi t}}\, e^{-q_1^2/2t} > 0$$ can reach origin

w_t^- is also called the <u>absorbing</u> Wiener density on $\mathbb{R}^+$

as $$w_t^-(0,q_1) = 0$$

Let $H_i := \frac{P^2}{2} + V(Q)$ act on $L^2(\mathbb{R}^+)$ with $i = D, N$ Dirichlet / Neumann condition at $r=0$

$$\rightsquigarrow \quad \langle q_b | e^{-tH_D} | q_a \rangle = \frac{q_a}{q_b} \int_{\mathcal{C}(\mathbb{R}^+; q_a)} dB^{(+\frac{1}{2})}[x]\, \delta(x(t)-q_b) \exp\left\{-\int_0^t d\tau\, V(x(\tau))\right\}$$

$$\langle q_b | e^{-tH_N} | q_a \rangle = \int_{\mathcal{C}(\mathbb{R}^+; q_a)} dB^{(-\frac{1}{2})}[x]\, \delta(x(t)-q_b) \exp\left\{-\int_0^t d\tau\, V(x(\tau))\right\}$$