Problem 12:

$$K(\varphi'',\varphi',t) = \frac{1}{2\pi}\sum_l e^{il(\varphi''-\varphi')} e^{-\frac{i\hbar t}{2mR^2}(l-\phi/\phi_0)^2}$$

$$= \frac{1}{2\pi}\sum_l e^{2\pi i z l} e^{i\pi\tau(l-\phi/\phi_0)^2}$$

with $z := \frac{\varphi''-\varphi'}{2\pi}$, $\tau := -\frac{\hbar t}{2mR^2\pi}$

$$= \frac{1}{2\pi} e^{i\pi\tau\frac{\phi^2}{\phi_0^2}} \sum_l \underbrace{e^{2\pi i z l} e^{-2\pi i\tau\frac{\phi}{\phi_0}l}}_{\exp\{2\pi i l(z-\tau\frac{\phi}{\phi_0})\}} e^{i\pi\tau l^2} =$$

$$= \frac{1}{2\pi} e^{i\pi\tau\phi^2/\phi_0^2}\, \Theta\left(z-\tau\frac{\phi}{\phi_0}\middle|\tau\right) \qquad \tilde z := z-\tau\frac{\phi}{\phi_0}$$

$$= \frac{1}{2\pi} e^{i\pi\tau\phi^2/\phi_0^2}\sqrt{\frac{i}{\tau}}\, e^{-i\pi\tilde z^2/\tau}\, \Theta\left(\frac{\tilde z}{\tau}\middle|-\frac{1}{\tau}\right)$$

$$= \frac{1}{2\pi}\exp\left\{i\pi\tau\left(\frac{\phi^2}{\phi_0^2}-\frac{\tilde z^2}{\tau^2}\right)\right\}\sqrt{\frac{i}{\tau}}\sum_n \exp\left\{i\pi n^2\left(-\frac{1}{\tau}\right)+2\pi i n\frac{\tilde z}{\tau}\right\}$$

$$= \frac{1}{2\pi}\exp\left\{i\pi\tau\left(\cancel{\frac{\phi^2}{\phi_0^2}}-\frac{z^2}{\tau^2}+2\frac{z}{\tau}\frac{\phi}{\phi_0}-\cancel{\frac{\phi^2}{\phi_0^2}}\right)\right\}\sqrt{\frac{i}{\tau}}\sum_n \exp\left\{-\frac{i\pi}{\tau}(n^2-2n\tilde z)\right\}$$

$$= \frac{1}{2\pi}\sqrt{\frac{i}{\tau}}\exp\left\{i\pi\tau\left(\underset{5}{2\frac{z}{\tau}\frac{\phi}{\phi_0}}-\underset{4}{\frac{z^2}{\tau^2}}\right)\right\}\sum_n \exp\left\{-\frac{i\pi}{\tau}\left(\underset{2}{n^2}-\underset{3}{2nz}+\underset{1}{2n\tau\frac{\phi}{\phi_0}}\right)\right\}$$

$$= \frac{1}{2\pi}\sqrt{\frac{i}{\tau}}\sum_n \exp\left\{\underset{1}{-2\pi i n\frac{\phi}{\phi_0}}\right\}\exp\left\{-\frac{i\pi}{\tau}\left(\underset{2}{n^2}-\underset{3}{2nz}+\underset{4}{z^2}\right)+\underset{5}{2\pi i z\frac{\phi}{\phi_0}}\right\}$$

$$= \frac{1}{2\pi}\sqrt{\frac{i}{\tau}}\sum_n \exp\left\{i(\varphi''-\varphi'-2\pi n)\frac{\phi}{\phi_0}\right\}\exp\left\{-i\frac{\pi}{\tau}(n-z)^2\right\}$$

$$\underset{n\to -n}{=} \sum_n \sqrt{\frac{mR^2}{2\pi i\hbar t}}\, e^{i(\varphi''-\varphi'+2\pi n)\frac{\phi}{\phi_0}} \exp\left\{\frac{i}{\hbar}\frac{mR^2}{2t}(\varphi''-\varphi'+2\pi n)^2\right\}$$

Problem 13: $z := \hbar\omega\beta$ small, $\omega^2 = \frac{1}{m} V''(x)$

$$\rho(x,x,\beta) = \sqrt{\frac{m\omega}{2\pi\hbar \sinh z}} \exp\left\{-\frac{m\omega}{\hbar}\frac{V'^2}{V''}\frac{\cosh z - 1}{\sinh z}\right\} e^{-\beta\left(V - \frac{V'^2}{2V''}\right)}$$

$$\sinh z \approx z + \frac{1}{3!}z^3 = z\left(1 + \frac{z^2}{6}\right) \quad \rightsquigarrow \sqrt{\frac{1}{\sinh z}} \approx \frac{1}{\sqrt{z}}\left(1 - \frac{z^2}{12}\right)$$

$$\frac{\cosh z - 1}{\sinh z} \approx \frac{\frac{1}{2}z^2 + \frac{1}{4!}z^4}{z\left(1 + z^2/6\right)} \approx \frac{z}{2}\,\frac{1 + z^2/12}{1 + z^2/6} \approx \frac{z}{2}\left(1 - z^2/12\right)$$

$$\rho(x,x,\beta) = \sqrt{\frac{m\omega}{2\pi\hbar z}}\left(1 - \frac{z^2}{12}\right)\exp\left\{-\frac{m\omega}{\hbar}\frac{V'^2}{V''}\frac{z}{2}\left(1 - \frac{z^2}{12}\right)\right\} e^{-\beta\left(V - \frac{V'^2}{2V''}\right)}$$

$$= \sqrt{\frac{m}{2\pi\hbar^2\beta}}\left(1 - \frac{z^2}{12}\right)\underbrace{\exp\left\{-\frac{m\omega^2}{2\hbar}\hbar\beta\frac{V'^2}{V''^2}\left(1 - \frac{z^2}{12}\right)\right\}}_{\exp\left\{-\frac{\beta}{2}\frac{V'^2}{V''}\left(1 - \frac{z^2}{12}\right)\right\}} e^{-\beta\left(V - \frac{V'^2}{2V''}\right)}$$

$$= \sqrt{\frac{m}{2\pi\hbar^2\beta}}\, e^{-\beta V}\left(1 - \frac{z^2}{12}\right)\exp\left\{+\frac{\beta}{2}\frac{V'^2}{V''}\frac{z^2}{12}\right\}$$

$$\approx \sqrt{\frac{m}{2\pi\hbar^2\beta}}\, e^{-\beta V}\underbrace{\left(1 - \frac{z^2}{12}\right)\left(1 + \frac{\beta V'^2}{24 V''}z^2\right)}_{1 - \frac{\hbar^2\beta^2\omega^2}{12} + \frac{\beta V'^2}{24}\frac{\hbar^2\beta^2\omega^2}{} = 1 - \frac{\hbar^2\beta^2}{12m}\left(V''(x) - \frac{\beta}{2}V'^2(x)\right)}$$

$$= \sqrt{\frac{m}{2\pi\hbar^2\beta}}\, e^{-\beta V(x)}\left(1 + \frac{\hbar^2\beta^2}{24m}\left(\beta V'^2(x) - 2V''(x)\right)\right)$$

with $\int dx\, e^{-\beta V(x)} V''(x) = -\int dx\,(-\beta V')\, e^{-\beta V} V' = \int d\beta\, \beta V'^2 e^{-\beta V}$

Partition function:

$$Z(\beta) = \int dx\, \rho(x, x, \beta)$$

$$\approx \int dx \sqrt{\frac{m}{2\pi\hbar^2\beta}}\, e^{-\beta V(x)} \left(1 - \frac{\hbar^2\beta^3}{24m} V'^2(x)\right)$$

or

$$\approx \int dx \sqrt{\frac{m}{2\pi\hbar^2\beta}}\, e^{-\beta V(x)} \left(1 - \frac{\hbar^2\beta^2}{24m} V''(x)\right)$$

$$\rightsquigarrow \quad Z(\beta) = \sqrt{\frac{m}{2\pi\hbar^2\beta}} \int dx\, e^{-\beta V_{eff}(x)}$$

with

$$V_{eff}(x) = V(x) + \frac{\hbar^2\beta^2}{24m} V'^2(x) = V(x) + \frac{\hbar^2\beta}{24m} V''(x)$$

See Feynman-Hibbs
Problem 10-2