3. Homework

Problem 8: Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\partial_q^2 + V(q) - E\right)\phi(q) = 0$$

Let $q = f(x) \rightsquigarrow \partial q = \frac{\partial x}{\partial q}\partial x = \frac{1}{f'(x)}\partial_x$, $\quad \phi(q) = h(x)\varphi(x)$

$$\partial q\,\phi(q) = \frac{1}{f'(x)}\partial_x h(x)\varphi(x) = \frac{h'}{f'}\varphi + \frac{h}{f'}\varphi' = \frac{1}{f'}(h'\varphi + h\varphi')$$

$$\partial_q^2\phi(q) = \frac{1}{f'}\partial_x\left(\frac{1}{f'}(h'\varphi + h\varphi')\right) =$$

$$= \frac{1}{f'}\left[-\frac{f''}{f'^2}(h'\varphi + h\varphi') + \frac{1}{f'}(h''\varphi + h'\varphi' + h'\varphi' + h\varphi'')\right]$$

$$= \frac{h}{f'}\varphi'' + \frac{2h'f' - hf''}{f'^3}\varphi' + \frac{f'h'' - f''h'}{f'^3}\varphi$$

Form invariance of SE $\rightarrow$ φ'-Term to vanish

$$\rightsquigarrow \quad 2h'f' = hf'' \rightsquigarrow \frac{h'}{h} = \frac{1}{2}\frac{f''}{f'} \rightsquigarrow \ln h = \frac{1}{2}\ln f' + \text{const}$$

$$\Rightarrow \quad h(x) = C\sqrt{f'(x)} \quad \text{with } C > 0, \quad C = e^{\text{const}}$$

Let $C \equiv 1$: $\quad h'(x) = \frac{1}{2}\frac{f''(x)}{\sqrt{f'(x)}}, \quad h''(x) = -\frac{1}{4}\frac{f''^2}{f'^{3/2}} + \frac{f'''}{\sqrt{f'}}\frac{1}{2}$

$$\partial_q^2\phi = \frac{h}{f'^2}\left(\varphi'' + \left(\frac{h''}{h} - \frac{h'f''}{hf'}\right)\varphi\right)$$

$$\frac{h''}{h} = -\frac{1}{4}\frac{f''^2}{f'^2} + \frac{1}{2}\frac{f'''}{f'} \quad , \quad \frac{h'}{h}\frac{f''}{f'} = \frac{1}{2}\frac{f''^2}{f'^2}$$ (2)

$$\rightsquigarrow \quad \partial_q^2 \Phi = \frac{h}{f'^2}\left(\varphi'' + \frac{1}{2}\frac{f'''}{f'} - \frac{3}{4}\frac{f''^2}{f'^2}\varphi\right)$$

With Schwarz derivative: $\boxed{(Sf)(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2}$

$$\rightsquigarrow \quad \partial_q^2 \Phi = \frac{h}{f'^2}\left(\varphi'' + \frac{1}{2}(Sf)(x)\,\varphi(x)\right)$$

insert in Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{h}{f'^2}\left(\varphi'' + \frac{1}{2}(Sf)\,\varphi(x)\right) + \left(V(f(x)) - E\right) h\, \varphi(x) = 0$$

$$\rightsquigarrow \quad \frac{h(x)}{f'^2(x)}\left[-\frac{\hbar^2}{2m}\partial_x^2 + f'^2(x)\left(V(f(x)) - E\right) + \frac{\hbar^2}{4m}(Sf)(x)\right] = 0$$

$$\Rightarrow \quad \underline{\left[-\frac{\hbar^2}{2m}\partial_x^2 + \tilde{V}(x) - \tilde{E}\right]\varphi(x) = 0}$$

where $\tilde{V}(x) := f'^2(x)\left[V(f(x)) - E\right] - \frac{\hbar^2}{4m}(Sf)(x) + \tilde{V}_0$

$\tilde{E} := \tilde{V}_0$ trivially added to have form-invariance

$(V, E) \longrightarrow (\tilde{V}, \hat{E})$

Problem 9: set $q \equiv r$ and $f(x) = e^x$ ($\nearrow$ Problem 8)

$$\rightsquigarrow f = f' = f'' = f''' = e^x \rightsquigarrow (Sf)(x) = 1 - \frac{3}{2} = -\frac{1}{2}$$

$$\rightsquigarrow \tilde{V}(x) = e^{2x}\left[V(e^x) + \frac{\hbar^2 l(l+1)}{2m\, e^{2x}} - E\right] + \hat{V}_0 + \frac{\hbar^2}{8m}$$

$$= [V(e^x) - E]\, e^{2x} + \frac{\hbar^2}{2m}\underbrace{\left(l(l+1) + \frac{1}{4}\right)}_{(l+\frac{1}{2})^2} + \tilde{V}_0$$

$$\rightsquigarrow \left[-\frac{\hbar^2}{2m}\partial_x^2 + [V(e^x) - E]\, e^{2x} + \frac{\hbar^2}{2m}(l+\tfrac{1}{2})^2\right]\varphi(x) = 0 \quad SE^*$$

#

WKB-Formula: For $H = \frac{p^2}{2m} + V(Q)$

semi-classical quantisation condition, $V(q_{L/R}) = E$

$$\int_{q_L}^{q_R} dq \sqrt{2m\,(E - V(q))} = \hbar\pi\,(n+\tfrac{1}{2})$$

In the original SE we have $V_{eff}(r) = V(r) + \frac{\hbar^2}{2m} l(l+1)$, $q \equiv r$

$$\rightsquigarrow \int_{r_L}^{r_R} dr\ 2m\left(E - V(r) - \frac{\hbar^2}{2mr^2} l(l+1)\right)^{1/2} = \hbar\pi\,(n+\tfrac{1}{2})$$

$$\rightsquigarrow \underline{\lambda^2 = l(l+1)}$$

In transformed SE* we have $q = x$ with $V_{eff}(x) = \tilde{V}(x)$

$$\rightsquigarrow \int_{x_L}^{x_R} dx \sqrt{2m(\tilde{E} - V_{eff}(x))} = \hbar\pi (n + \tfrac{1}{2})$$

Note $\tilde{E} - V_{eff}(x) = \left(E - V(e^x)\right)e^{2x} - \frac{\hbar^2}{2m}(l+\tfrac{1}{2})^2$

Let $r := e^x \rightsquigarrow dx = \frac{1}{r}dr$

$$\rightsquigarrow \int_{r_L}^{r_R} dr \frac{1}{r}\left[2m(E - V(r))r^2 - \frac{\hbar^2}{2m}(l+\tfrac{1}{2})^2\right]^{1/2} =$$

$$\int_{r_L}^{r_R} dr \left[2m\left(E - V(r) - \frac{\hbar^2}{2mr^2}(l+\tfrac{1}{2})^2\right)\right]^{1/2} \rightsquigarrow \lambda^2 = (l+\tfrac{1}{2})^2$$

Langer-transformation:

Replace in radial WKB formula $l(l+1) \rightarrow (l+\tfrac{1}{2})^2$

Example: $V(r) = \frac{m}{2}\omega^2 r^2$ radial harmonic oscillator

$$V_{eff}(r) = \frac{m}{2}\omega^2 r^2 + \frac{\hbar^2}{2mr^2}\lambda^2, \quad V_{eff}(r_{L/R}) := E$$

Calculate

$$\int_{r_L}^{r_R} dr \left[2m\left(E - \frac{m}{2}\omega^2 r^2 - \frac{\hbar^2}{2m}\frac{\lambda^2}{r^2}\right)\right]^{1/2}$$

$$I := \int_{r_L}^{r_R} dr \left[2mE - m^2\omega^2 r^2 - \frac{\hbar^2\lambda^2}{r^2} \right]^{1/2} \underset{x=r^2}{=} \frac{1}{2} \int_{x_L}^{x_R} \frac{dx}{x} \left[2mEx - m^2\omega^2 x^2 - \hbar^2\lambda^2 \right]^{1/2}$$

$$= \frac{m\omega}{2} \int_{x_L}^{x_R} \frac{dx}{x} \left(bx - x^2 - c \right)^{1/2} \quad \text{with} \quad b := \frac{2E}{m\omega^2}, \quad c := \frac{\hbar^2\lambda^2}{m^2\omega^2}$$

$$x_{L/R} = -\frac{1}{2}\left(-b \pm \sqrt{b^2 - 4c}\right) = \frac{b}{2} \mp \sqrt{\frac{b^2}{4} - c}$$

$$I = \frac{m\omega}{2} \underbrace{\int_{x_L}^{x_R} \frac{dx}{x} \sqrt{(x_R - x)(x - x_L)}}_{= \frac{\pi}{2}(x_R + x_L) - \pi\sqrt{x_R x_L}}$$

$$= \frac{m\omega}{2} \left[\frac{\pi}{2} b - \pi\sqrt{c} \right] = \frac{m\omega\pi}{2} \left[\frac{E}{m\omega^2} - \frac{\hbar\lambda}{m\omega} \right] = \frac{\pi\hbar}{2} \left[\frac{E}{\hbar\omega} - \lambda \right]$$

$$I \overset{!}{=} \hbar\pi(n + \tfrac{1}{2})$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{E}{\hbar\omega} - \lambda \right] = n + \frac{1}{2} \quad \Leftrightarrow \quad \frac{E}{\hbar\omega} = 2n + 1 + \lambda$$

$$E = \hbar\omega(2n + \lambda + 1)$$

for $\lambda = l + \frac{1}{2}$

$$\underline{\underline{E = \hbar\omega\left(2n + l + \tfrac{3}{2}\right)}}$$

GR 2.267:

$$\int dx\,\frac{\sqrt{R}}{x} = \sqrt{R} + a\int\frac{dx}{x\sqrt{R}} + \frac{b}{2}\int\frac{dx}{\sqrt{R}}$$

$$R = (x_R - x)(x - x_L) = -x_L x_R + x(x_R + x_L) - x^2$$

$$\Rightarrow \quad a = -x_R x_L \quad b = x_R + x_L \quad c = -1, \quad \Delta = 4ac - b^2 = -(x_R - x_L)^2$$

$$\sqrt{R}\Big|_{x_L}^{x_R} = 0 \quad , \quad \int\frac{dx}{x\sqrt{R}} = \frac{1}{\sqrt{-a}}\arcsin\frac{2a+bx}{x\sqrt{-\Delta}} \qquad \text{GR 2.266}$$

$$\int\frac{dx}{\sqrt{R}} = -\frac{1}{\sqrt{-c}}\arcsin\frac{2cx+b}{\sqrt{-\Delta}} \qquad \text{GR 2.261}$$

$$\rightsquigarrow \int_{x_L}^{x_R}\frac{dx}{x}\sqrt{(x_R-x)(x-x_L)} = \frac{-x_L x_R}{\sqrt{x_L x_R}}\arcsin\frac{x(x_L+x_R)-2x_L x_R}{x(x_L+x_R)}\Bigg|_{x_L}^{x_R} - \frac{x_L+x_R}{2}\arcsin\frac{x_R-x_L-2x}{x_R+x_L}$$

$$= -\sqrt{x_L x_R}\Big(\underbrace{\arcsin 1}_{\frac{\pi}{2}} - \underbrace{\arcsin(-1)}_{+\frac{\pi}{2}}\Big) - \frac{x_L+x_R}{2}\Big(\underbrace{\arcsin(-1)}_{-\frac{\pi}{2}} - \underbrace{\arcsin(1)}_{-\frac{\pi}{2}}\Big)$$

$$= \pi\left(\frac{x_L+x_R}{2} - \sqrt{x_L x_R}\right)$$

B. Homework cont.

Problem 10: $$f(x) := \sum_{n\in\mathbb{Z}} g(x+n)$$

a) $$f(x+1) = \sum_n g(x+1+n) \underset{n'=n+1}{=} \sum_{n'} g(x+n') = f(x) \quad \#$$

Fourier Series for f:

$$f(x) = \sum_{m\in\mathbb{Z}} c_m\, e^{2\pi i m x}$$

$$c_m = \int_0^1 dx\, f(x)\, e^{-2\pi i m x} = \sum_{n\in\mathbb{Z}} \int_0^1 dx\, g(x+n)\, e^{-2\pi i m x}$$

$$\overset{x_n = x+n}{=} \sum_n \int_n^{n+1} dx_n\, g(x_n)\, e^{-2\pi i m x_n} = \int_{-\infty}^{+\infty} dx\, g(x)\, e^{-2\pi i m x} \quad \#$$

b) $$f(x) = \sum_n g(x+n) = \sum_m c_m\, e^{2\pi i m x}$$

$$= \sum_m \int_{-\infty}^{+\infty} dy\, g(y)\, e^{2\pi i m (x-y)}$$

$$\rightsquigarrow \quad \sum_n g(x+n) = \sum_m \int_{-\infty}^{+\infty} dy\, g(y)\, e^{2\pi i m (x-y)}$$

Poisson formula

let $g(z) = \delta(z)$

$$\rightsquigarrow \quad \sum_{n\in\mathbb{Z}} \delta(x-n) = \sum_{m\in\mathbb{Z}} e^{2\pi i m x}$$

also known as Poisson formula

(2)

c) Let $g(x) = e^{-ax^2}$, $\mathrm{Re}\, a > 0$

$$\curvearrowright \int_{-\infty}^{+\infty} dy\; e^{-ay^2} e^{-2\pi i m y} = \sqrt{\frac{\pi}{a}}\; \exp\left\{-\frac{(2\pi m)^2}{4a}\right\}$$

$$\Rightarrow \boxed{\sum_{n\in\mathbb{Z}} e^{-a(x+n)^2} = \sqrt{\frac{\pi}{a}} \sum_{m\in\mathbb{Z}} e^{-\frac{\pi^2 m^2}{a}} e^{2\pi i m x}} \quad (*)$$

Problem 11:

Let $\boxed{\Theta(z,\tau) := \sum_{n\in\mathbb{Z}} \exp\{i\pi\tau n^2 + 2\pi i n z\}}$, $z \in \mathbb{C}$, $\mathrm{Im}\,\tau > 0$ $(**)$

a) Use $a = \frac{i\pi}{\tau}$ in $(*)$

RHS of $(*)$: $\sqrt{\frac{\tau}{i}} \sum_m \exp\{i\pi\tau m^2 + 2\pi i m z\} = \sqrt{\frac{\tau}{i}}\, \Theta(z|\tau)$

$$\curvearrowright \Theta(z|\tau) = \sqrt{\frac{i}{\tau}} \sum_n \exp\left\{-\frac{i\pi}{\tau}(z+n)^2\right\}$$

b)

- $\Theta(z+1|\tau) = \Theta(z|\tau)$ obvious as $e^{2\pi i n(z+1)} = e^{2\pi i n z}$ in definition $(**)$

- $\Theta(-z|\tau) = \Theta(z|\tau)$ obvious as $z \to -z$ and $n \to -n$ in $(**)$

- $\Theta(z+\tau|\tau) = \sum_n \exp\{i\pi\tau n^2 + 2\pi i n(z+\tau)\} = \sum_n \exp\{i\pi\tau(n^2+2n) + 2\pi i n z\}$

$$= \sum_n \exp\{i\pi\tau(n+1)^2 - i\pi\tau + 2\pi i(n+1)z - 2\pi i z\}$$

$$= e^{-i\pi\tau} e^{-2\pi i z}\, \Theta(z|\tau)$$

- $\Theta(z|\tau) = 0$ for $z = \frac{1}{2} + \frac{\tau}{2} + n + m\tau$

Sufficient to show for $n=0$ and $m=0$ as rest follow from 1.+3. property!

$$\Theta\left(\tfrac{1}{2}+\tfrac{\tau}{2}\middle|\tau\right) = \sum_n \exp\left\{i\pi\tau n^2 + 2\pi i n\left(\tfrac{1}{2}+\tfrac{\tau}{2}\right)\right\}$$

$$= \sum_n \exp\left\{i\pi\tau(n^2+n)\right\}(-1)^n$$

$$= \sum_{n=0}^{\infty} \exp\left\{i\pi\tau(n^2+n)\right\}(-1)^n + \sum_{n=-1}^{-\infty} \exp\left\{i\pi\tau(n^2+n)\right\}(-1)^n$$

let $m = -n-1 \rightsquigarrow m^2+m = (n+1)^2 - n - 1 = n^2+n$

and $(-1)^m = -(-1)^n$

$$= \sum_{n=0}^{\infty} \exp\left\{i\pi\tau(n^2+n)\right\}(-1)^n - \sum_{m=0}^{\infty} \exp\left\{i\pi\tau(m^2+m)\right\}(-1)^m = 0 \;\#$$

- From a)

$$\Theta(z|\tau) = \sqrt{\frac{i}{\tau}} \sum_n \exp\left\{-\frac{i\pi}{\tau}(z+n)^2\right\}$$

$$= \sqrt{\frac{i}{\tau}}\, e^{-i\frac{\pi z^2}{\tau}} \sum_n e^{-\frac{i\pi}{\tau}n^2} e^{-2\pi i n z/\tau}$$

$$= \sqrt{\frac{i}{\tau}}\, e^{-i\frac{\pi z^2}{\tau}} \sum_n e^{i\pi\left(-\frac{1}{\tau}\right)n^2} e^{2\pi i n\left(-\frac{z}{\tau}\right)}$$

$$= \sqrt{\frac{i}{\tau}}\, e^{-i\frac{\pi z^2}{\tau}} \overset{\text{2. Property}}{\Theta\left(+\frac{z}{\tau}\middle|-\frac{1}{\tau}\right)}$$

$$\Rightarrow \Theta\left(\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\, e^{i\pi z^2/\tau}\, \Theta(z|\tau) \;\#$$

$$\theta_3(z,t) := \Theta\left(\tfrac{z}{\pi}\middle| t\right)$$

$$\rightsquigarrow \quad \theta_3(z+\pi,t) = \Theta\left(\tfrac{z+\pi}{\pi}\middle| t\right) = \Theta\left(\tfrac{z}{\pi}\middle| t\right) = \theta_3(z,t)$$

$$\theta_3(-z,t) = \theta_3(z,t)$$

$$\theta_3(z+\pi t,t) = \Theta\left(\tfrac{z}{\pi}+t\middle| t\right) = e^{-i\pi t}\, e^{-2iz}\, \theta_3(z,t)$$

$$\theta_3(z,t)=0 \rightsquigarrow \frac{z}{\pi} = \frac{1}{2}+\frac{t}{2}+n+mt$$

$$\rightsquigarrow \quad z = \frac{\pi}{2}+\frac{\pi t}{2}+n\pi+m\pi t$$

$$\theta_3(z,t) = \Theta\left(\tfrac{z}{\pi}\middle| t\right) = \sqrt{\frac{i}{t}}\, e^{-iz^2/\pi t}\, \Theta\left(\tfrac{z}{\pi t}\middle| -\tfrac{1}{t}\right)$$

$$= \sqrt{\frac{i}{t}}\, e^{-iz^2/\pi t}\, \theta_3\left(\frac{z}{t}, -\frac{1}{t}\right)$$