2. Homework (↗ B. Felsager, "Geometry, Particles and Fields", p. 173f)

Problem 6: $L(\dot{q},q) = \frac{m}{2}\dot{q}^2 - \frac{m}{2}\omega^2 q^2$

$\rightsquigarrow$ Euler-Lagrange-Eq.: $\ddot{q} - \omega^2 q = 0$

a) General solution: $x(\tau) = a\cos\omega\tau + b\sin\omega\tau$, $a,b \in \mathbb{R}$

as obviously $\ddot{x}(\tau) = -\omega^2 x(\tau)$ #

Let $x(0) = x' \rightsquigarrow \underline{\underline{a = x'}}$

$x(t) = x'' \rightsquigarrow x'' = x'\cos\omega t + b\sin\omega t$

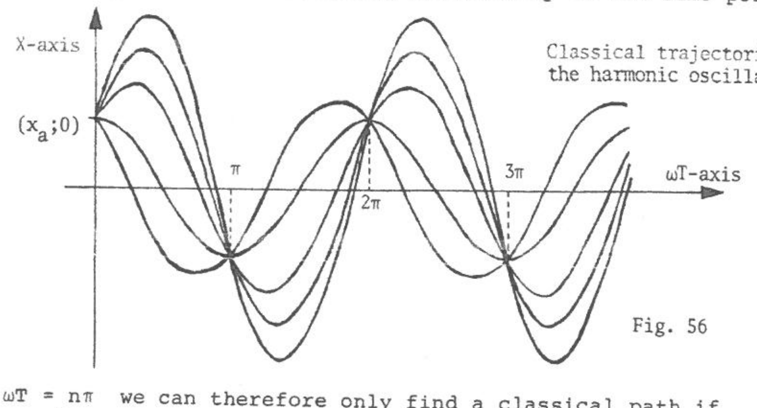
$\sin\omega t \neq 0 \Leftrightarrow \omega t \neq n\pi$, $n \in \mathbb{N}$

$$\rightsquigarrow \quad \underline{\underline{b = \frac{x'' - x'\cos\omega t}{\sin\omega t}}}$$

Let $t = \frac{n\pi}{\omega} \rightsquigarrow x'' = x(t) = x'\cos n\pi = (-1)^n x'$

x'' is <u>not</u> arbitrary for $\omega t = n\pi$

If on the contrary $\omega T = n\pi$ (corresponding to either a half period or a full period) we are in trouble. This is because a classical particle, after a half-period, necessarily is in the opposite point. After a full period it is similarly necessarily in the same point.



Classical trajectories for the harmonic oscillator.

Fig. 56

For $\omega T = n\pi$ we can therefore only find a classical path if $x_b = (-1)^n x_a$, and, if that is the case, *any* classical path passing through $(x_a;0)$ will in fact also pass through $(x_b;T)$! In analogy with optics we say that the classical path has a *caustic* when $\omega T = n\pi$. At the caustics the propagator thus becomes singular!

from Felsager

b) The classical action

$$S_{cl}(x'',x',t) = \int_0^t d\tau \left[\frac{m}{2}\dot{x}^2(\tau) - \frac{m}{2}\omega^2 x^2(\tau) \right]$$

$$\frac{m}{2}\left[\dot{x}^2 - \omega^2 x^2\right] = \frac{m}{2}\omega^2\left[(-a\sin\omega\tau + b\cos\omega\tau)^2 - (a\cos\omega\tau + b\sin\omega\tau)^2\right]$$

$$= \frac{m}{2}\omega^2\left[a^2\sin^2\omega\tau + b^2\cos^2\omega\tau - 2ab\cos\omega\tau\sin\omega\tau - a^2\cos^2\omega\tau - b^2\sin^2\omega\tau - 2ab\cos\omega\tau\sin\omega\tau\right]$$

$$= \frac{m}{2}\omega^2\left[(b^2-a^2)\underbrace{(\cos^2\omega\tau - \sin^2\omega\tau)}_{\cos 2\omega\tau} - 4ab\underbrace{\cos\omega\tau\sin\omega\tau}_{\frac{1}{2}\sin 2\omega\tau}\right]$$

$$\leadsto S_{cl} = \frac{m}{2}\omega^2\int_0^t d\tau\left[(b^2-a^2)\cos 2\omega\tau - 2ab\sin 2\omega\tau\right]$$

$$\int_0^t d\tau\cos 2\omega\tau = \frac{1}{2\omega}\sin 2\omega\tau\Big|_0^t = \frac{\sin 2\omega t}{2\omega} = \frac{1}{\omega}\sin\omega t\cos\omega t$$

$$\int_0^t d\tau\sin 2\omega\tau = -\frac{1}{2\omega}\cos 2\omega\tau\Big|_0^t = \frac{1}{2\omega}(1-\cos 2\omega t) = \frac{1}{\omega}\sin^2\omega t$$

$$\leadsto S_{cl} = \frac{m\omega}{2}\left[(b^2-a^2)\sin\omega t\cos\omega t - 2ab\sin^2\omega t\right]$$

$$(b^2-a^2)\cos\omega t\sin\omega t = \left(\frac{x''^2 - 2x''x'\cos\omega t + x'^2\cos^2\omega t}{\sin^2\omega t} - x'^2\right)\cos\omega t\sin\omega t$$

$$= x''^2\frac{\cos\omega t}{\sin\omega t} + x'^2\frac{\cos\omega t}{\sin\omega t}(\cos^2\omega t - \sin^2\omega t) - 2x''x'\frac{\cos^2\omega t}{\sin\omega t}$$

$$2ab\sin^2\omega t = 2\sin\omega t(x''x' - x'^2\cos\omega t) = 2x''x'\sin\omega t - 2x'^2\sin\omega t\cos\omega t$$

$$S_{cl} = \frac{m\omega}{2}\left[x''^2\frac{\cos\omega t}{\sin\omega t} + x'^2\frac{\cos\omega t}{\sin\omega t}\underbrace{(\cos^2\omega t - \sin^2\omega t + 2\sin^2\omega t)}_{=1} - 2x''x'\frac{1}{\sin\omega t}\underbrace{(\cos^2\omega t + \sin^2\omega t)}_{=1}\right]$$

$$S_{cl} = \frac{m\omega}{2\sin\omega t}\left[(x''^2 + x'^2)\cos\omega t - 2x''x'\right]$$

③

<u>Problem 7:</u> $H(P,Q) = \frac{P^2}{2m} + \frac{m}{2}\omega^2 Q^2$, $\mathcal{H} = L^2(\mathbb{R})$

$$H|\varphi_n\rangle = E_n |\varphi_n\rangle$$

with $E_n = \hbar\omega(n+\frac{1}{2})$ $\quad n = 0,1,2,3,\ldots$

$$\langle x|\varphi_n\rangle = C_n \exp\left\{-\frac{x^2}{2\lambda^2}\right\} H_n(x/\lambda)$$

with $H_n(z)$ = Hermite polynomial

$$\lambda := \sqrt{\frac{\hbar}{m\omega}}, \quad C_n := \sqrt{\frac{1}{2^n n!}} \frac{1}{\sqrt{\lambda\sqrt{\pi}}}$$

Spectral representation of propagator

$$K(x'',x',t) = \sum_{n=0}^{\infty} \exp\left\{-\frac{i}{\hbar}E_n t\right\} \varphi_n(x'')\varphi_n^*(x')$$

$$x := x''/\lambda, \quad y := x'/\lambda$$

$$\curvearrowright K(x'',x',t) = \frac{1}{\lambda\sqrt{\pi}} e^{-\frac{1}{2}(x^2+y^2)} \sum_{n=0}^{\infty} \frac{1}{2^n n!} e^{-i\omega t(n+\frac{1}{2})} H_n(x) H_n(y)$$

let $\varsigma := e^{-i\omega t}$ $\quad (\omega t \neq N\pi)$ $N \in \mathbb{N}$

$$= \frac{1}{\lambda\sqrt{\pi}} e^{-\frac{1}{2}(x^2+y^2)} \sqrt{\varsigma} \underbrace{\sum_{n=0}^{\infty} \left(\frac{\varsigma}{2}\right)^n \frac{1}{n!} H_n(x) H_n(y)}_{\text{Mehler Formula}}$$

$$= \frac{1}{\lambda\sqrt{\pi}} e^{-\frac{1}{2}(x^2+y^2)} \sqrt{\varsigma} \frac{1}{\sqrt{1-\varsigma^2}} \exp\left\{-\frac{\varsigma^2(x^2+y^2) - 2\varsigma xy}{1-\varsigma^2}\right\}$$

- $$\sqrt{\frac{\varsigma}{1-\varsigma^2}} = \sqrt{\frac{e^{-i\omega t}}{1-e^{-2i\omega t}}} = \sqrt{\frac{1}{e^{i\omega t} - e^{-i\omega t}}} = \sqrt{\frac{1}{2i\sin\omega t}}$$

$$\Leftrightarrow \quad \frac{1}{\lambda\sqrt{\pi}}\sqrt{\frac{s}{1-s^2}} = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega t}}$$

- $$\frac{s^2(x^2+y^2)}{1-s^2} + \frac{1}{2}(x^2+y^2) = (x^2+y^2)\frac{2s^2+1-s^2}{2(1-s^2)} =$$
$$= (x^2+y^2)\frac{1}{2}\frac{1+s^2}{1-s^2} = \frac{x^2+y^2}{2}\frac{e^{i\omega t}+e^{-i\omega t}}{e^{i\omega t}-e^{-i\omega t}} = \underline{\frac{\cos\omega t}{i\sin\omega t}}\frac{x^2+y^2}{2}$$

- $$\frac{2sxy}{1-s^2} = xy\frac{2e^{-i\omega t}}{1-e^{-2i\omega t}} = xy\frac{1}{i\sin\omega t}$$

$\Rightarrow$

$$K(x'',x',t) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega t}}\exp\left\{\frac{i}{2}(x^2+y^2)\frac{\cos\omega t}{\sin\omega t} - \frac{ixy}{\sin\omega t}\right\}$$
$$= \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega t}}\exp\left\{\frac{im\omega}{2\hbar\sin\omega t}(x''^2+x'^2)\cos\omega t - \frac{im\omega}{\hbar\sin\omega t}x''x'\right\}$$
$$\underline{\underline{= \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega t}}\exp\left\{\frac{i}{\hbar}S_{cl}(x'',x',t)\right\}}}$$

Let $\omega t = N\pi \quad \Rightarrow \quad e^{-\frac{i}{\hbar}E_n t} = \exp\{-iN\pi(n+\frac{1}{2})\} = e^{-i\frac{N}{2}\pi}\left(e^{-iN\pi}\right)^n$

N odd: $K(x'',x',\frac{N\pi}{\omega}) = \sum_{n=0}^{\infty} e^{-\frac{i}{2}N\pi}(-1)^n \varphi_n(x)\varphi_n^*(y)$, $\varphi_n(-y) = (-1)^n\varphi(y)$

$$= e^{-\frac{i}{2}N\pi}\sum_{n=0}^{\infty}\varphi_n(x)\varphi_n^*(-y) = \underline{\underline{e^{-\frac{i}{2}N\pi}\delta(x+y)}}$$

N even:

$K(x'',x',\frac{N\pi}{\omega}) = \underline{\underline{e^{-\frac{i}{2}N\pi}\delta(x-y)}}$ caustics ?