1. Homework

Problem 1a: $\quad I_0 := \int\limits_{-\infty}^{+\infty} dx\, e^{-ax^2} \quad , a > 0$

Consider $\quad I_0^2 = \int\limits_{-\infty}^{+\infty} dx \int\limits_{-\infty}^{+\infty} dy\; e^{-a(x^2+y^2)} \underset{\text{Polar Coordinates}}{=} 2\pi \int\limits_0^{\infty} dr\, r e^{-ar^2} \underset{t=r^2}{=}$

$$= \pi \int\limits_0^{\infty} dt\, e^{-at} = \pi \left[-\frac{1}{a} e^{-at} \right]_0^{\infty} = \frac{\pi}{a} \quad \rightsquigarrow \quad \underline{\underline{I_0 = \sqrt{\frac{\pi}{a}}}} \quad \#$$

Problem 1b: $\quad$ Consider $\; -ax^2 + bx = -a\left(x^2 - \frac{b}{a}x\right) = -a\left(x - \frac{b}{2a}\right)^2 + \frac{b^2}{4a}$

$$\rightsquigarrow \int\limits_{\mathbb{R}} dx\, \exp\{-ax^2 + bx\} = \int\limits_{\mathbb{R}} dx\, \exp\left\{-a\left(x - \frac{b}{2a}\right)^2\right\} e^{\frac{b^2}{4a}} = \underline{\underline{\sqrt{\frac{\pi}{a}}\, e^{b^2/4a}}}$$

$a > 0, \; b \in \mathbb{R}$ or even $b \in \mathbb{C}$

Problem 1c:

$$I_n := \int\limits_{\mathbb{R}} dx\, x^{2n} e^{-ax^2} = \sqrt{\frac{\pi}{a}}\, \frac{(2n-1)!!}{(2a)^n} \qquad n = 1, 2, 3, \ldots$$

$$I_1 = \int\limits_{\mathbb{R}} dx\, x^2 e^{-ax^2} = -\frac{\partial}{\partial a} \int\limits_{\mathbb{R}} dx\, e^{-ax^2} = -\frac{\partial}{\partial a} I_0 = -\frac{\partial}{\partial a} \sqrt{\frac{\pi}{a}}$$

$$= -\sqrt{\pi}\left(-\frac{1}{2} a^{-\frac{3}{2}}\right) = \sqrt{\frac{\pi}{a}}\, \frac{1}{2a} \qquad n=1 \text{ o.k.}$$

$$I_{n+1} = -\frac{\partial}{\partial a} I_n = -\frac{\partial}{\partial a} \frac{\sqrt{\pi}\,(2n-1)!!}{2^n a^{n+1/2}} = \frac{\sqrt{\pi}\,(2n-1)!!}{2^n} \left(-\frac{\partial}{\partial a} a^{-n-\frac{1}{2}}\right)$$

$$= \frac{\sqrt{\pi}\,(2n-1)!!}{2^n} \left(n + \frac{1}{2}\right) a^{-n-\frac{3}{2}} = \frac{\sqrt{\pi}\,(2n-1)!!}{2^n}\, \frac{(2n+1)}{2a^{n+1}\sqrt{a}}$$

$$= \sqrt{\frac{\pi}{a}}\, \frac{(2n+1)!!}{(2a)^{n+1}}$$

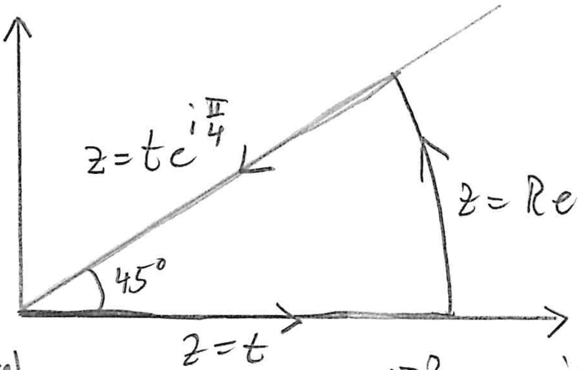
Problem 2:

Follows formally from 1a by replacing $a \to -ia$

$$\leadsto \int_{\mathbb{R}} dx\, e^{iax^2} = \sqrt{\frac{\pi}{-ia}} = \sqrt{\frac{\pi}{|a|}} \begin{cases} e^{-i\pi/4} & \text{for } a<0 \\ e^{i\pi/4} & a>0 \end{cases}$$

More rigorous: Consider for $a>0$: $\oint_{\mathcal{C}} dz\, e^{iaz^2} = 0$

with contour $\mathcal{C}$:



$$\leadsto 0 = \oint_{\mathcal{C}} dz\, e^{iaz} =$$

$$= \int_0^\infty dt\, e^{iat^2} + \int_0^{\pi/4} d\varphi\, (iRe^{i\varphi}) \underbrace{\exp\{iaR^2 e^{2i\varphi}\}} + e^{i\frac{\pi}{4}} \int_{-\infty}^{0} dt\, e^{iat^2 e^{i\frac{\pi}{2}}}$$

$$\exp\{iaR^2(\cos 2\varphi + i \sin 2\varphi)\}$$

$$= \exp\{-aR^2 \underbrace{\sin 2\varphi}_{>0 \text{ for } 0<\varphi<\frac{\pi}{4}}\} e^{iaR^2 \cos 2\varphi}$$

$$\longrightarrow 0 \text{ for } R \to \infty$$

$$e^{i\frac{\pi}{4}} \int_{-\infty}^{0} dt\, e^{-at^2} = -\sqrt{\frac{\pi}{a}}\, \frac{e^{i\frac{\pi}{4}}}{2}$$

$a>0$!

$$\Rightarrow \int_{-\infty}^{\infty} dt\, e^{iat^2} = \sqrt{\frac{\pi}{a}}\, e^{i\pi/4} \quad \#$$

For $a<0$ consider $\oint_{\mathcal{C}} dz\, e^{az^2}$ with same contour

$$\leadsto \int_{-\infty}^{+\infty} dt\, e^{iat^2} = \sqrt{\frac{\pi}{|a|}}\, e^{-i\frac{\pi}{4}} \quad \#$$

Problem 3 : A positive and symmetric $\rightsquigarrow$ positive eigenvalues $a_i > 0$ and $\exists$ orthogonal matrix O such that $O^T A O = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$

$\rightsquigarrow$ $\vec{y} = O^T \vec{x}$ $\qquad \dfrac{\partial(\vec{y})}{\partial(\vec{x})} = \det O = 1$

$$\rightsquigarrow \int_{\mathbb{R}^n} d^n\vec{x} \, \exp\{-\tfrac{1}{2} \vec{x}^T A \vec{x}\} = \int_{\mathbb{R}^n} d^n\vec{y} \, \exp\{-\tfrac{1}{2} \vec{y}^T (O^T A O) \vec{y}\}$$

$$= \prod_{i=1}^n \int_{\mathbb{R}} dy_i \, e^{-\frac{1}{2} y_i a_i y_i} = \prod_{i=1}^n \sqrt{\left(\frac{2\pi}{a_i}\right)} = \boxed{\sqrt{\frac{(2\pi)^n}{\det A}}}$$

Problem 4 : A symmetric $\rightsquigarrow$ as in problem 3 $\exists$ O

rank $A = n$ $\rightsquigarrow$ all eigenvalues are real and non-zero $a_i \in \mathbb{R}/0$

$$\rightsquigarrow \int_{\mathbb{R}^n} d^n\vec{x} \, \exp\{\tfrac{i}{2} \vec{x}^T A \vec{x}\} = \prod_{i=1}^n \int_{\mathbb{R}} dy_i \, e^{\frac{i}{2} a_i y_i^2}$$

m : number of negative eigenvalues (Morse-Index)

$\rightsquigarrow$ $n-m$: number of positive eigenvalues

$\rightsquigarrow$ use Problem 2 $\Rightarrow$

$$\int_{\mathbb{R}^n} d^n\vec{x} \, \exp\{\tfrac{i}{2} \vec{x}^T A \vec{x}\} = \prod_{i=1}^n \sqrt{\frac{2\pi}{|a_i|}} \; e^{i\frac{\pi}{4}(n-m)} \, e^{-i\frac{\pi}{4} m}$$

$$= \sqrt{\frac{(2\pi i)^n}{|\det A|}} \; e^{-im\frac{\pi}{2}}$$

Problem 5:

Consider
$$\int dx\, K(x'', x, \tau_1)\, K(x, x', \tau_2) =$$

$$= \int dx \sqrt{\frac{m}{2\pi i\hbar \tau_1}} \exp\left\{\frac{im}{2\hbar\tau_1}(x''-x)^2\right\} \sqrt{\frac{m}{2\pi i\hbar\tau_2}} \exp\left\{\frac{im}{2\hbar\tau_2}(x-x')^2\right\}$$

$$= \frac{m}{2\pi i\hbar} \frac{1}{\sqrt{\tau_1\tau_2}} \exp\left\{\frac{im}{2\hbar}\left(\frac{x''^2}{\tau_1} + \frac{x'^2}{\tau_2}\right)\right\} \underbrace{\int dx \exp\left\{\frac{im}{2\hbar}\left(\frac{x^2-2xx''}{\tau_1} + \frac{x^2-2xx'}{\tau_2}\right)\right\}}_{=:I}$$

$$I = \int dx \exp\left\{\underbrace{\frac{im}{2\hbar}\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)}_{=-a} x^2 - \underbrace{\frac{im}{\hbar}\left(\frac{x''}{\tau_1}+\frac{x'}{\tau_2}\right)}_{=-b} x\right\}$$ of Problem 1b

$$= \sqrt{\frac{\pi\, 2i\hbar}{m(\tau_1+\tau_2)}\tau_1\tau_2}\; \exp\left\{-\frac{m^2}{\hbar^2}\left(\frac{x''}{\tau_1}+\frac{x'}{\tau_2}\right)^2 \frac{2i\hbar\tau_1\tau_2}{4(\tau_1+\tau_2)m}\right\}$$

$$= \sqrt{\frac{2\pi i\hbar}{m(\tau_1+\tau_2)}}\sqrt{\tau_1\tau_2}\; \exp\left\{-\frac{im}{2\hbar}\frac{(x''\tau_2+x'\tau_1)^2}{\tau_1\tau_2(\tau_1+\tau_2)}\right\}$$

$$* = \sqrt{\frac{m}{2\pi i\hbar(\tau_1+\tau_2)}} \exp\left\{\frac{im}{2\hbar}\frac{1}{\tau_1\tau_2}\underbrace{\left(x''^2\tau_2 + x'^2\tau_1 - \frac{x''^2\tau_2^2 + x'^2\tau_1^2 + 2x''x'\tau_1\tau_2}{\tau_1+\tau_2}\right)}_{=A}\right\}$$

$$A = \frac{1}{\tau_1+\tau_2}\left(x''^2\tau_2(\tau_1+\cancel{\tau_2}) + x'^2\tau_1(\cancel{\tau_1}+\tau_2) - \cancel{x''^2\tau_2^2} - \cancel{x'^2\tau_1^2} - 2x''x'\tau_1\tau_2\right)$$

$$= \frac{1}{\tau_1+\tau_2}\left(x''^2\tau_1\tau_2 + x'^2\tau_1\tau_2 - 2x''x'\tau_1\tau_2\right) = \frac{\tau_1\tau_2}{\tau_1+\tau_2}(x''-x')^2$$

$$* = \sqrt{\frac{m}{2\pi i\hbar(\tau_1+\tau_2)}} \exp\left\{\frac{im}{2\hbar}\frac{(x''-x')^2}{(\tau_1+\tau_2)}\right\}$$ #