6. Homework in "Path Integrals"

Problem 17: The Wiener process on the real line $\mathcal{Q} = \mathbb{R}$

Show that the transition density for the Wiener process on the real line

$$w_t(x_2, x_1) := \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(x_2 - x_1)^2}{2t}\right\}, \qquad t \geq 0, \qquad x_1, x_2 \in \mathbb{R},$$

obeys the condition for a continuous stationary Markov process:

- Positivity: $\quad w_t(x_2, x_1) \geq 0$
- Normalisation: $\quad \displaystyle\int_{-\infty}^{\infty} \mathrm{d}x_2\, w_t(x_2, x_1) = 1$
- Initial Condition: $\quad \displaystyle\lim_{t \searrow 0} w_t(x_2, x_1) = \delta(x_2 - x_1)$
- Chapman-Kolmogorov: $\quad \displaystyle\int_{-\infty}^{\infty} \mathrm{d}x_2\, w_t(x_3, x_2) w_t(x_2, x_1) = w_t(x_3, x_1)$
- Lindeberg: $\quad \displaystyle\lim_{t \searrow 0} \frac{1}{t} \int_{|x_2 - x_1| > \varepsilon} \mathrm{d}x_2\, w_t(x_2, x_1) = 0.$

  You may use: $\displaystyle\mathrm{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^{\infty} \mathrm{d}y\, \mathrm{e}^{-y^2} = \frac{\mathrm{e}^{-z^2}}{z\sqrt{\pi}} \left(1 + O(z^{-2})\right).$

Problem 18: Markov processes on the positive half-line $\mathcal{Q} = \mathbb{R}^+$

On the positive half line $\mathbb{R}^+$ one may define a reflecting and absorbing Wiener process, respectively, via

$$w_t^{\pm}(x_2, x_1) := w_t(x_2, x_1) \pm w_t(x_2, -x_1), \qquad t \geq 0, \qquad x_1, x_2 \in \mathbb{R}^+.$$

a) Do these transition densities obey the conditions for a continuous stationary Markov process given in above Problem 17?

b) The Bessel process with index ν is a continuous stationary Markov process on the positive half line with transition density given by

$$b_t^{(\nu)}(x_2, x_1) := \frac{x_2}{t} \left(\frac{x_2}{x_1}\right)^{\nu} \exp\left\{-\frac{x_2^2 + x_1^2}{2t}\right\} \mathrm{I}_\nu\left(\frac{x_1 x_2}{t}\right), \qquad t \geq 0, \qquad x_1, x_2 \in \mathbb{R}^+,$$

Here $\mathrm{I}_\nu(z)$ denotes the modified Bessel function of first kind with index ν.

Show that the reflecting and absorbing Wiener densities defined above are related to the Bessel process densities as follows

$$w_t^{+}(x_2, x_1) = b_t^{(-\frac{1}{2})}(x_2, x_1), \qquad w_t^{-}(x_2, x_1) = \frac{x_1}{x_2} b_t^{(\frac{1}{2})}(x_2, x_1)$$

Hint: $\mathrm{I}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sinh z, \qquad \mathrm{I}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cosh z,$