4. Homework in "Path Integrals"

Problem 12: Winding number representation of idealised Aharonov-Bohm setup

The eigenfunctions and eigenvalues of a charged particle of mass $m > 0$ and electric charge e moving on a ring of radius $R > 0$ around a solenoid with magnetic flux Φ are given by

$$\psi_\ell(\varphi) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}\ell\varphi}\,, \qquad E_\ell = \frac{\hbar^2}{2mR^2}\left(\ell - \frac{\Phi}{\Phi_0}\right)^2, \qquad \Phi_0 = 2\pi\frac{\hbar c}{e}\,, \qquad \ell \in \mathbb{Z}\,.$$

Starting from the spectral representation of the propagator, derive its winding number representation as shown below

$$\begin{aligned} K(\varphi'', \varphi'; t) &= \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\ell(\varphi''-\varphi')}\,\exp\left\{-\frac{\mathrm{i}\hbar t}{2mR^2}\left(\ell - \frac{\Phi}{\Phi_0}\right)^2\right\} \\ &= \sum_{n\in\mathbb{Z}} \exp\left\{\mathrm{i}(\varphi'' - \varphi' + 2\pi n)\frac{\Phi}{\Phi_0}\right\} K_n(\varphi'', \varphi'; t)\,, \end{aligned}$$

where the free partial propagator for n windings is given by

$$K_n(\varphi'', \varphi'; t) := \sqrt{\frac{mR^2}{2\pi\mathrm{i}\hbar t}}\,\exp\left\{\frac{\mathrm{i}}{\hbar}\frac{mR^2}{2t}(\varphi'' - \varphi' + 2\pi n)^2\right\}.$$

Problem 13: Wigner-Kirkwood expansion

As shown in the lecture, the quasi-classical approximation of the diagonal density matrix for a one-dimensional particle in an external potential $V = V(x)$ is given by

$$\rho_\beta(x, x) = \mathrm{e}^{-\beta\left(V - \frac{1}{2}\frac{V'^2}{V''}\right)}\sqrt{\frac{m\omega}{2\pi\hbar\sinh(\hbar\omega\beta)}}\,\exp\left\{-\frac{m\omega}{\hbar}\left(\frac{V'}{V''}\right)^2\frac{\cosh(\hbar\omega\beta) - 1}{\sinh(\hbar\omega\beta)}\right\},$$

where $\omega^2 = V''(x)/m$. Show that for small $\hbar$ the partition function

$$Z(\beta) = \int_{-\infty}^{\infty} \mathrm{d}x\,\rho_\beta(x, x)$$

can be written as

$$Z(\beta) = \sqrt{\frac{m}{2\pi\hbar^2\beta}}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{e}^{-\beta V_{\text{eff}}(x)}$$

with effective quantum potential

$$V_{\text{eff}}(x) = V(x) - \frac{\hbar^2\beta}{24m}\,V''(x) + O(\hbar^4)\,.$$

Hint:

Assuming $\lim_{x\to\pm\infty} V(x) = +\infty$, integration by parts gives

$$\int_{-\infty}^{\infty} \mathrm{d}x\,V''(x)\mathrm{e}^{-\beta V(x)} = \beta\int_{-\infty}^{\infty} \mathrm{d}x\,V'^2(x)\,\mathrm{e}^{-\beta V(x)}$$