3. Homework in "Path Integrals"

Problem 8: Change of Variables in the Stationary Schrödinger Equation

Consider the one-dimensional stationary Schrödinger equation with scalar potential V

$$\left(-\frac{\hbar^2}{2m}\,\partial_q^2 + V(q) - E\right)\Phi(q) = 0.$$

Show that the change of independent variable $q \to x$ defined by $q =: f(x) \in C^\infty$ and the change of the depended variable $\Phi \to \phi$ defined by $\Phi(f(x)) =: h(x)\phi(x)$ leads to a new stationary Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\,\partial_x^2 + \tilde{V}(x) - \tilde{E}\right)\phi(x) = 0,$$

if one sets $h(x) = \sqrt{f'(x)}$, $\tilde{E} = \tilde{V}_0 \in \mathbb{R}$ and

$$\tilde{V}(x) := f'^2(x)\,[V(f(x)) - E] + \tilde{V}_0 - \frac{\hbar^2}{4m}(Sf)(x)$$

with Schwarz-Derivative

$$(Sf)(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Problem 9: The Langer Transformation

Show that the radial Schrödinger equation appearing for centrally symmetric potentials $V(\mathbf{r}) = V(r)$, $r = |\mathbf{r}| \in \mathbb{R}^+$, that is

$$\left(-\frac{\hbar^2}{2m}\,\partial_r^2 + V(r) + \frac{\hbar^2 l(l+1)}{2mr^2} - E\right)\Phi(r) = 0,$$

can be transformed into a problem on the real line $x \in \mathbb{R}$ via the transformation $r =: \mathrm{e}^x$ resulting in

$$\left(-\frac{\hbar^2}{2m}\,\partial_x^2 + [V(\mathrm{e}^x) - E]\,\mathrm{e}^{2x} + \frac{\hbar^2(l+1/2)^2}{2m}\right)\phi(x) = 0.$$

Show that the WKB quantisation formula (see tutorial exercise 6) for the latter equation reads

$$\int_{r_L}^{r_R} \mathrm{d}r \sqrt{2m\left(E - V(r) - \frac{\hbar^2(l+1/2)^2}{2mr^2}\right)} = \hbar\pi(n+1/2).$$

That is, in the original centrifugal potential the term $l(l+1)$ is replaced by $(l+1/2)^2$. This so-called Langer modification of the WKB formula results in exact energy eigenvalues of the radial harmonic oscillator and the Coulomb problem. See for example A. Galindo and P. Pascual, Quantum Mechanics II.

Problem 10: Poisson Summation Formula

Consider an arbitrary function g defined on the real line and define a related function f

$$f(x) := \sum_{n\in\mathbb{Z}} g(x+n).$$

a) Show that $f(x+1) = f(x)$ and its Fourier series representation can be written as

$$f(x) = \sum_{m\in\mathbb{Z}} c_m\,\mathrm{e}^{2\pi\mathrm{i}mx}\,, \quad c_m = \int_{-\infty}^{\infty} \mathrm{d}x\, g(x)\,\mathrm{e}^{-2\pi\mathrm{i}mx}\,.$$

b) Show from this the identity (Poisson Summation Formula)

$$\sum_{n\in\mathbb{Z}} g(x+n) = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{d}y\, g(y)\,\mathrm{e}^{2\pi\mathrm{i}m(x-y)}$$

c) Specify in above Poisson summation formula $g(x) = \exp\{-ax^2\}$, $\mathrm{Re}\, a > 0$, and derive the relation

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-a(x+n)^2} = \sqrt{\frac{\pi}{a}} \sum_{m\in\mathbb{Z}} \exp\left\{-\pi^2 m^2/a + 2\pi\mathrm{i}mx\right\}$$

Problem 11: Jacobi's Theta Function

Jacobi's Theta Function is defined by

$$\Theta(z|\tau) := \sum_{n\in\mathbb{Z}} \exp\{\mathrm{i}\pi\tau n^2 + 2\pi\mathrm{i}nz\}\,, \quad z\in\mathbb{C}\,, \mathrm{Im}\,\tau > 0\,.$$

a) Show with the help of above result 10 c) the relation

$$\Theta(z|\tau) = \sqrt{\frac{\mathrm{i}}{\tau}} \sum_{n\in\mathbb{Z}} \exp\{-\mathrm{i}\pi(z+n)^2/\tau\}$$

b) Proof following properties of the Theta function

$\Theta(z+1|\tau) = \Theta(z|\tau)$

$\Theta(-z|\tau) = \Theta(z|\tau)$

$\Theta(z+\tau|\tau) = \mathrm{e}^{-\mathrm{i}\pi\tau}\mathrm{e}^{-2\pi\mathrm{i}z}\Theta(z|\tau)$

$\Theta(z|\tau) = 0$ for $z = \frac{1}{2} + \frac{\tau}{2} + n + m\tau$ where $n, m \in \mathbb{Z}$

$\Theta(\frac{z}{\tau}| -\frac{1}{\tau}) = \sqrt{\frac{\tau}{\mathrm{i}}}\,\mathrm{e}^{\mathrm{i}\pi z^2/\tau}\Theta(z|\tau)$

In the physics literature (see e.g. Schulman's book) the theta functions is sometimes defined by $\theta_3(z,t) := \Theta(\frac{z}{\pi}|t)$. Transfer above properties of $\Theta(z|\tau)$ into relations for $\theta_3(z,t)$.