2. Homework in "Path Integrals"

Problem 6: The Classical Action of the Harmonic Oscillator

The one-dimensional ($q \in \mathbb{R}$) classical dynamics of a harmonic oscillator with mass $m > 0$ and frequency $\omega > 0$ is characterized by the Lagrangian

$$L(\dot{q}, q) := \frac{m}{2}\,\dot{q}^2 - \frac{m}{2}\,\omega^2 q^2 \,.$$

a) Show that the general classical solution is given by

$$x_{cl}(\tau) = a\cos(\omega\tau) + b\sin(\omega\tau)$$

and for the boundary conditions $x_{cl}(0) = x'$ and $x_{cl}(t) = x''$ ($t > 0$) the constants a and b are given by

$$a = x'\,, \quad b = \frac{x'' - x'\cos(\omega\, t)}{\sin(\omega t)}\,, \quad \omega t \neq n\pi\,, \quad n \in \mathbb{N}\,.$$

b) Show by explicit calculation ($\omega t \neq n\pi$, $n \in \mathbb{N}$)

$$S_{cl}(x'', x', t) := \int_0^t \mathrm{d}\tau\, L\left(\dot{x}_{cl}(\tau), x_{cl}(\tau)\right) = \frac{m\omega}{2\sin(\omega t)}\left[\left(x''^2 + x'^2\right)\cos(\omega t) - 2x''x'\right]$$

Problem 7: The Quantum Propagator of the Harmonic Oscillator

The quantum dynamics of the one-dimensional harmonic oscillator (mass $m > 0$, frequency $\omega > 0$) is characterized by the Hamiltonian

$$H(P, Q) := \frac{P^2}{2m} + \frac{m}{2}\,\omega^2 Q^2 \quad \text{acting on} \quad \mathcal{H} = L^2(\mathbb{R})$$

with eigenvalues E_n and eigenstates $|\varphi_n\rangle$, $H|\varphi_n\rangle = E_n|\varphi_n\rangle$, given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)\,, \quad \langle x|\varphi_n\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left\{-\frac{m\omega}{2\hbar}x^2\right\} H_n\left(x\sqrt{m\omega/\hbar}\right)$$

where $n \in \mathbb{N}_0$ and H_n denotes the Hermite polynomial of order n.

Use the spectral representation of the propagator $K(x'', x', t) = \langle x''|\exp\{-\mathrm{i}Ht/\hbar\}|x'\rangle$ and the Mehler formula

$$\frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{\rho^2(x^2+y^2) - 2\rho xy}{1-\rho^2}\right\} = \sum_{n=0}^{\infty}\frac{(\rho/2)^n}{n!}H_n(x)H_n(y)\,, \quad \rho \in \mathbb{C}\backslash\{-1, 1\}\,,$$

to show that (Hint: $\rho = \mathrm{e}^{\mathrm{i}\omega t}$)

$$K(x'', x', t) = \sqrt{\frac{m\omega}{2\pi\mathrm{i}\hbar\sin(\omega t)}}\exp\left\{(\mathrm{i}/\hbar)S_{cl}(x'', x', t)\right\}\,, \quad (\omega t/\pi) \notin \mathbb{N}_0\,.$$

Discuss the case $(\omega t/\pi) \in \mathbb{N}_0$ using the spectral representation of the propagator.