1. Homework in "Path Integrals"

Problem 1: Gauß-Integrals:

Proof the following integral formulas for $a, b \in \mathbb{R}$, $a > 0$:

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} &= \sqrt{\frac{\pi}{a}} \\ \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2+bx} &= \sqrt{\frac{\pi}{a}} \exp\left\{\frac{b^2}{4a}\right\} \\ \int_{-\infty}^{+\infty} \mathrm{d}x\, x^{2n}\mathrm{e}^{-ax^2} &= \sqrt{\frac{\pi}{a}} \frac{(2n-1)!!}{(2a)^n}\,,\end{aligned}$$

where $n \in \mathbb{N}_0$, $(2n-1)!! := 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$.

Problem 2: Fresnel-Integral:

Proof the following integral formula for $a \in \mathbb{R}\backslash\{0\}$

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}ax^2} = \sqrt{\frac{\pi}{|a|}} \begin{cases} \mathrm{e}^{-\mathrm{i}\pi/4} & \text{for} \quad a < 0 \\ \mathrm{e}^{+\mathrm{i}\pi/4} & \text{for} \quad a > 0 \end{cases}\,.$$

Problem 3: Multi-dimensional Gauß-Integral:

Let A be a symmetric positive definite $n \times n$ matrix. Show that

$$\int_{\mathbb{R}^n} \mathrm{d}^n\vec{x}\, \exp\left\{-\frac{1}{2}\,\vec{x}^{\mathrm{T}} A\vec{x}\right\} = \sqrt{\frac{(2\pi)^n}{\det A}}\,.$$

Problem 4: Multi-dimensional Fresnel-Integral:

Let A be a symmetric $n \times n$ matrix of rank n, i.e. it has only non-zero eigenvalues. Show that

$$\int_{\mathbb{R}^n} \mathrm{d}^n\vec{x}\, \exp\left\{\frac{\mathrm{i}}{2}\,\vec{x}^{\mathrm{T}} A\vec{x}\right\} = \sqrt{\frac{(2\pi\mathrm{i})^n}{|\det A|}}\, \mathrm{e}^{-\mathrm{i}m\pi/2}\,,$$

where the Morse-Index m is the number of negative eigenvalues of A.

Problem 5: Convolution of the Free-Particle Propagator:

Show by explicit integration that the free-particle propagator

$$K_0(x'', x', t) := \sqrt{\frac{m}{2\pi\mathrm{i}\hbar t}} \exp\left\{\frac{\mathrm{i}m}{2\hbar t}(x'' - x')^2\right\}$$

obeys the convolution property

$$\int_{-\infty}^{\infty} \mathrm{d}x\, K_0(x'', x, t'' - t) K_0(x, x', t - t') = K_0(x'', x', t'' - t')$$