Exercise 15: The SUSY Foldy-Wouthuysen-Transformation

Given: $Q_1 = \begin{pmatrix} 0 & A \\ A^+ & 0 \end{pmatrix}$, $\mathcal{M} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}$, $W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$H_D = Q_1 + \mathcal{M} W = \begin{pmatrix} M_+ & A \\ A^+ & -M_- \end{pmatrix} \quad \text{with } AM_- = M_+ A \text{ and } A^+ M_+ = M_- A^+$$

Claim: $$H_D^{FW} := U_{FW} H_D U_{FW}^+ = \begin{pmatrix} \sqrt{AA^+ + M_+^2} & 0 \\ 0 & -\sqrt{A^+A + M_-^2} \end{pmatrix}$$

where $U_{FW} := a_+ + W \operatorname{sgn} Q_1 \, a_-$, $a_\pm := \sqrt{\frac{1}{2} \pm \frac{\mathcal{M}}{2|H_D|}}$

Note:

- $\{Q_1, W\} = \begin{pmatrix} 0 & -A \\ A^+ & 0 \end{pmatrix} + \begin{pmatrix} 0 & A \\ -A^+ & 0 \end{pmatrix} = 0$
- $[\mathcal{M}, W] = \begin{pmatrix} M_+ & 0 \\ 0 & -M_- \end{pmatrix} - \begin{pmatrix} M_+ & 0 \\ 0 & -M_- \end{pmatrix} = 0$
- $[\mathcal{M}, Q_1] = \begin{pmatrix} 0 & M_+A \\ M_-A^+ & 0 \end{pmatrix} - \begin{pmatrix} 0 & AM_+ \\ A^+M_- & 0 \end{pmatrix} = 0$
- $H_D^2 = Q_1^2 + Q_1 MW + MWQ_1 + MWMW = Q_1^2 + \mathcal{M}^2 = \begin{pmatrix} AA^+ + M_+^2 & 0 \\ 0 & A^+A + M_-^2 \end{pmatrix}$

  $\rightsquigarrow [H_D^2, W] = [Q_1^2, W] = Q_1^2 W - W Q_1^2 = 0$

  $\rightsquigarrow [|H_D|, W] = 0 = [\sqrt{Q_1^2}, W]$

- $\operatorname{sgn} Q_1 = \frac{Q_1}{\sqrt{Q_1^2}} \rightsquigarrow \{\operatorname{sgn} Q_1, W\} = 0$

Lemma: $H_D W \operatorname{sgn} Q_1 = (Q_1 + \mathcal{M}W) W \operatorname{sgn} Q_1 = Q_1 W \operatorname{sgn} Q_1 + \mathcal{M}W^2 \operatorname{sgn} Q_1$

$= -W Q_1 \operatorname{sgn} Q_1 - \mathcal{M} W \operatorname{sgn} Q_1 W$

$= -W \operatorname{sgn} Q_1 Q_1 - W \operatorname{sgn} Q_1 \mathcal{M} W = -W \operatorname{sgn} Q_1 (Q_1 + \mathcal{M}W)$

$= -W \operatorname{sgn} Q_1 H_D$

• FW-Transformation:

$$U_{FW} := a_+ + W\,\mathrm{sgn}\,Q_1\, a_- \quad , \quad a_\pm := \sqrt{\frac{1}{2} \pm \frac{M}{2|H_D|}} = a_\pm^\dagger$$

$$\leadsto U_{FW}^\dagger = a_+^\dagger + a_-^\dagger (\mathrm{sgn}\,Q_1)^\dagger W^\dagger = a_+ + a_-\,\mathrm{sgn}\,Q_1\, W$$

$$= a_+ + \mathrm{sgn}\,Q_1\, W a_- = a_+ - W\,\mathrm{sgn}\,Q_1\, a_-$$

Unitarity: $U_{FW} U_{FW}^\dagger = (a_+ + W\,\mathrm{sgn}\,Q_1\, a_-)(a_+ - W\,\mathrm{sgn}\,Q_1\, a_-)$

$$= a_+^2 - W\,\mathrm{sgn}\,Q_1\, W\,\mathrm{sgn}\,Q_1\, a_-^2 = a_+^2 + a_-^2 = 1$$

Note: $a_\pm^2 = \frac{1}{2} \pm \frac{M}{2|H_D|} \leadsto a_+^2 + a_-^2 = 1$

$$a_+^2 - a_-^2 = \frac{M}{|H_D|}$$

and $2a_+a_- = 2\left(\frac{1}{4} - \frac{M^2}{4H_D^2}\right)^{1/2} = \sqrt{1 - \frac{M^2}{H_D^2}}$

$$= \sqrt{\frac{H_D^2 - M^2}{H_D^2}} = \sqrt{\frac{Q_1^2}{H_D^2}} = \frac{|Q_1|}{|H_D|}$$

• Transformation of H_D

$$U_{FW} H_D U_{FW}^\dagger = (a_+ + W\,\mathrm{sgn}\,Q_1\, a_-) H_D (a_+ - W\,\mathrm{sgn}\,Q_1\, a_-)$$

$$= (a_+ H_D + W\,\mathrm{sgn}\,Q_1\, a_- H_D)(a_+ - W\,\mathrm{sgn}\,Q_1\, a_-)$$

$$= a_+ H_D a_+ + W\,\mathrm{sgn}\,Q_1\, a_- H_D a_+ - a_+ H_D W\,\mathrm{sgn}\,Q_1\, a_- - W\,\mathrm{sgn}\,Q_1\, a_+ H_D W\,\mathrm{sgn}\,Q_1\, a_-$$

Lemma

$$= a_+^2 H_D + a_- a_+ W\,\mathrm{sgn}\,Q_1\, H_D + a_+ a_- W\,\mathrm{sgn}\,Q_1\, H_D + a_-^2 W\,\mathrm{sgn}\,Q_1\, W\,\mathrm{sgn}\,Q_1\, H_D$$

$$= (a_+^2 + 2a_- a_+ W\,\mathrm{sgn}\,Q_1 - a_-^2) H_D$$

$$= \left(\frac{M}{|H_D|} + W\,\mathrm{sgn}\,Q_1 \frac{|Q_1|}{|H_D|}\right) H_D = W\left(\frac{WM + Q_1}{|H_D|}\right) H_D$$

$$= W \frac{H_D}{|H_D|} H_D = W \frac{H_D^2}{|H_D|} = W|H_D| = W\sqrt{H_D^2} \quad \text{diagonal}$$

$$= \begin{pmatrix} \sqrt{AA^\dagger + M_+^2} & 0 \\ 0 & -\sqrt{A^\dagger A + M_-^2} \end{pmatrix} \quad \#$$

Exercise 16: Examples of SUSY Dirac Hamiltonians

• Electrons in a magnetic field

$$H_D = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot(\vec{p}-\frac{e}{c}\vec{A}) \\ c\vec{\sigma}\cdot(\vec{p}-\frac{e}{c}\vec{A}) & -mc^2 \end{pmatrix} \Rightarrow A = c\vec{\sigma}\cdot(\vec{p}-\frac{e}{c}\vec{A}) = A^+$$

$$M_\pm = mc^2$$

$$H_\pm = \frac{1}{2mc^2}\left\{\begin{matrix} AA^+ \\ A^+A \end{matrix}\right\} = \frac{1}{2m}(\vec{p}-\frac{e}{c}\vec{A})^2 - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}$$

Pauli Hamiltonian with $N=2$ SUSY

• Electron in a spherical tensor potential

$$A = c\vec{\sigma}\cdot(\vec{p} - i\vec{e}_r U'(r)) \quad , \; M_\pm = mc^2$$

$$\rightsquigarrow \quad H_\pm = \frac{\vec{p}^2}{2m} + \frac{1}{2m}U'^2(r) \pm \frac{1}{2m}U''(r) \pm \frac{U'(r)}{mr}K \quad \to \text{Pauli case?}$$

• Electron in a scalar field

in SUSY reps

$$H = \begin{pmatrix} 0 & c\vec{\sigma}\cdot\vec{p} - i(mc^2+\phi_{sc}) \\ c\vec{\sigma}\cdot\vec{p} + i(mc^2+\phi_{sc}) & 0 \end{pmatrix}$$

$$A = c(\vec{\sigma}\cdot\vec{p}) - i(mc^2+\phi_{sc})$$

$$M_\pm = 0$$

$$\rightsquigarrow \; AA^+ = c^2\vec{p}^2 + (\phi_{sc}+mc^2)^2 + c\hbar\,\vec{\sigma}\cdot(\vec{\nabla}\phi_{sc})$$

$$H_\pm = \frac{\vec{p}^2}{2m} + W^2(r) \pm \frac{\hbar}{\sqrt{2m}}\vec{\sigma}\cdot(\vec{\nabla}W(\vec{r}))$$

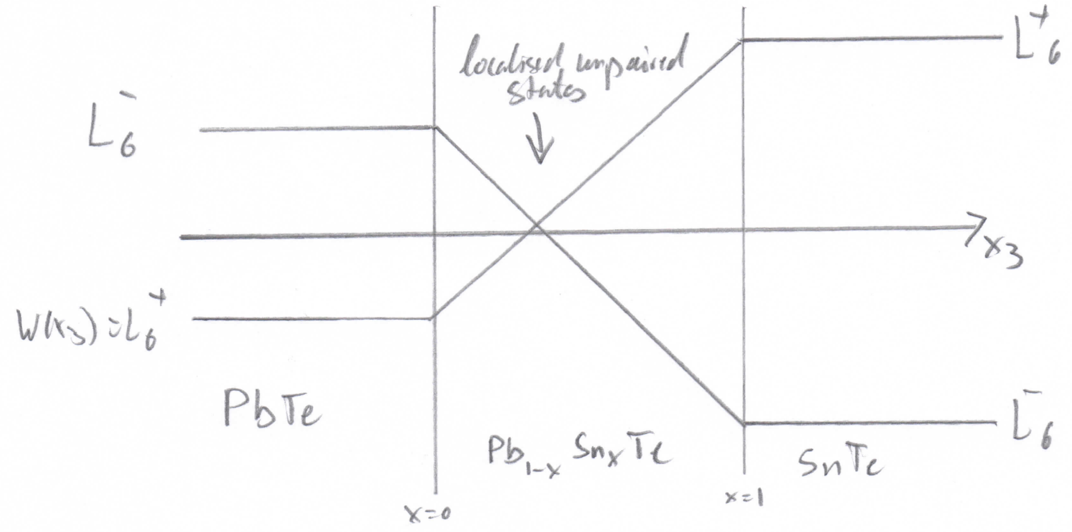
$$W(\vec{r}) := \frac{1}{\sqrt{2mc^2}}(\phi_{sc}(\vec{r}) + mc^2)$$

assume $\phi_{sc}(\vec{r}) = \phi_{sc}(x_3)$

$$\rightsquigarrow \; H_\pm = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \underbrace{\left[\frac{p_3^2}{2m} + W^2(x_3) \pm \frac{\hbar}{\sqrt{2m}}W'(x_3)\sigma_3\right]}_{\text{Pair of Witten Hamiltonians}}$$

Application:

Description of position dependent mass gap in semi conductors like band inversion (unbroken SUSY)



• Two-dimensional electrons in magnetic field

$$H_D^{(2)} = c\sigma_1\left(P_1 - \frac{e}{c}a_1\right) + c\sigma_2\left(P_2 - \frac{e}{c}a_2\right) + mc^2\sigma_3$$

with $\vec{B} = B(x_1,x_2)\vec{e}_3$, $B(x_1,x_2) = \partial_1 a_2 - \partial_2 a_1$

$$\curvearrowright \quad H_D^{(2)} = \begin{pmatrix} mc^2 & A \\ A^+ & -mc^2 \end{pmatrix} \qquad A := (cP_1 - ea_1) - i(cP_2 - ea_2)$$

cf Pauli in 2D!

$$\left[H_D^{(2)}\right]^2 = 2mc^2 H_P^{(2)} + m^2c^4$$

let $B(x_1,x_2) = B = \text{const}.$ $\quad \omega_c := \frac{|eB|}{mc}$ cyclotron frequency

$\curvearrowright$ Eigenvalues of $H_P^{(2)}$: $\varepsilon_n = \hbar\omega_c n$, $n \in \mathbb{N}_0$

Landau Levels

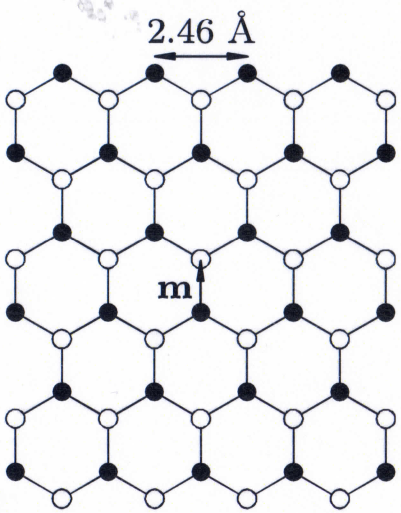
$\curvearrowright$ Eigenvalues of $H_D^{(2)}$:
$$E_n^{\pm} = \pm\sqrt{2mc^2\varepsilon_n + m^2c^4}$$
$$= \pm mc^2\sqrt{1 + 2n\frac{\hbar\omega_c}{mc^2}}$$

\#

same degeneracy d as for $H_P^{(2)}$

• Graphene

Graphene dispersion: 2D massless Dirac fermions



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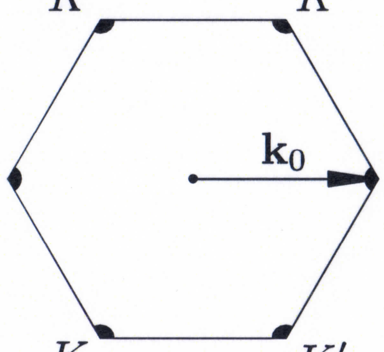

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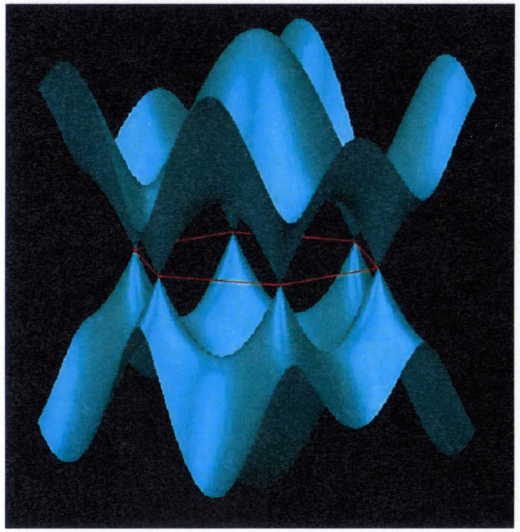

Two sublattices: A and B **Hamiltonian:** $H = \begin{pmatrix} 0 & t_{\mathbf{k}} \\ t_{\mathbf{k}}^* & 0 \end{pmatrix}$

$t_{\mathbf{k}} = t\left[1 + 2e^{i(\sqrt{3}/2)k_y a}\cos(k_x a/2)\right]$ **Spectrum** $\varepsilon_{\mathbf{k}}^2 = |t_{\mathbf{k}}|^2$

The gap vanishes at 2 points, $K, K' = (\pm k_0, 0)$, where $k_0 = 4\pi/3a$.

In the vicinity of K, K': massless Dirac-fermion Hamiltonian:

$$H_K = v_0(k_x\sigma_x + k_y\sigma_y), \qquad H_{K'} = v_0(-k_x\sigma_x + k_y\sigma_y)$$

$v_0 \simeq 10^8$ cm/s – effective "light velocity", sublattice space $\longrightarrow$ isospin

Graphene: Zero gap semi conductor characterised by "Dirac-cones" near K, K' edge

$$H = V_F (\sigma_1 p_1 + \sigma_2 p_2) \qquad \text{non-interacting charge carriers}$$

$\uparrow$ Fermi-velocity $\approx 10^6 \frac{m}{s}$

interaction may be accounted for with eff. mass m_{eff} + plus magnetic field

$$H^{(\pm)} = \begin{pmatrix} m_{eff} V_F^2 & A_\pm \\ A_\pm & -m_{eff} V_F^2 \end{pmatrix} \qquad A_\pm := V_F \left((p_1 - \tfrac{e}{c} a_1) \mp i (p_2 - \tfrac{e}{c} a_2) \right)$$

remember 2D Pauli exercises 14 $\pm$ cases

Here $(\pm)$ for K, K' edge

Constant magnetic field

$$E_n^\pm = \pm \sqrt{2 m_{eff} V_F^2 \varepsilon_n + m_{eff}^2 V_F^4} = \pm \hbar \omega_F \sqrt{n + \frac{m_{eff}^2 V_F^4}{\hbar^2 \omega_F^2}}$$

$$\omega_F := V_F \sqrt{2 \frac{|eB|}{\hbar c}}$$

Applications: Unconventional quantum Hall effect

magnetic confinement (Klein tunneling)

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