Exercise 12: The Aharonov-Casher Theorem

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We consider ground state of Pauli-Hamiltonian in 2D with orthogonal magnetic field

$$\vec{B}(\vec{x}) = B(\vec{x})\,\vec{e}_3 \;, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$ with bounded support

• Magnetic Scalar Field:

$$\varphi(\vec{x}) := \int_{\mathbb{R}^2} d^2\vec{x}'\, B(\vec{x}')\, \ln|\vec{x}-\vec{x}'|$$

has dimension of a magn. flux

with $\Delta \ln|\vec{x}-\vec{x}'| = 2\pi\,\delta(\vec{x}-\vec{x}')$ Recall $\ln|\vec{x}-\vec{x}'|$ is Green function in 2D

$$\Longrightarrow \quad \underline{\underline{\Delta\varphi(\vec{x}) = 2\pi\, B(\vec{x})}}$$

• Vector Potential:

$$\vec{A}(\vec{x}) := \begin{pmatrix} a_1(\vec{x}) \\ a_2(\vec{x}) \end{pmatrix} := \frac{1}{2\pi}\begin{pmatrix} -\partial_2\varphi(\vec{x}) \\ \partial_1\varphi(\vec{x}) \end{pmatrix}$$

$$\Longrightarrow \partial_1 a_2 - \partial_2 a_1 = (\partial_1^2 + \partial_2^2)\,\frac{1}{2\pi}\varphi(\vec{x}) = B(\vec{x}) \quad \#$$

• From Lecture:

$$g = \pm 2 \qquad \Phi_0 := 2\pi\frac{\hbar c}{|e|}$$

$$A = \frac{1}{\sqrt{2m}}\left[\left(P_1 - \frac{e}{c}a_1\right) \mp i\left(P_2 - \frac{e}{c}a_2\right)\right]$$

$$\overset{*}{=} \frac{\hbar}{\sqrt{2m}}\, e^{\pm \varphi/\Phi_0\, \mathrm{sgn}\,e}\left[\mp\partial_2 - i\partial_1\right] e^{\mp \varphi/\Phi_0\, \mathrm{sgn}\,e}$$

Proof of ⊛:

$$A \overset{?}{=} \frac{\hbar}{\sqrt{2m}}\, e^{\pm\frac{\varphi}{\phi_0}\operatorname{sgn} e} \left(\mp\partial_2 - i\partial_1\right) e^{\mp\frac{\varphi}{\phi_0}\operatorname{sgn} e}$$

$$= \frac{\hbar}{\sqrt{2m}}\, e^{\pm\frac{\varphi}{\phi_0}\operatorname{sgn} e} e^{\mp\frac{\varphi}{\phi_0}\operatorname{sgn} e}\left[+\frac{\operatorname{sgn} e}{\phi_0}(\partial_2\varphi) \mp \partial_2 \pm i\frac{\operatorname{sgn} e}{\phi_0}(\partial_1\varphi) - i\partial_1\right]$$

$$= \frac{\hbar}{\sqrt{2m}}\left[\frac{e}{2\pi\hbar c}(-2\pi a_1) \mp \partial_2 \pm \frac{e}{2\pi\hbar c} i(2\pi a_2) - i\partial_1\right]$$

$$= \frac{1}{\sqrt{2m}}\left[-\frac{e}{c}a_1 \mp \hbar\partial_2 \pm i\frac{e}{c}a_2 - i\hbar\partial_1\right]$$

$$= \frac{1}{\sqrt{2m}}\left[\left(p_1 - \frac{e}{c}a_1\right) \mp i\left(p_2 - \frac{e}{c}a_2\right)\right] \quad \#$$

- Consider $\vec{x}\cdot\vec{x}' = |\vec{x}||\vec{x}'|\cos\theta$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{|\vec{x}'|^l}{|\vec{x}|^{l+1}}P_l(\cos\theta) = \frac{1}{|\vec{x}|}\left(1+O\left(\tfrac{1}{|\vec{x}|}\right)\right)$$

$$\rightsquigarrow \ln|\vec{x}-\vec{x}'| = -\ln\frac{1}{|\vec{x}-\vec{x}'|} = -\ln\frac{1}{|\vec{x}|} - \ln\left(1+O\left(\tfrac{1}{|\vec{x}|}\right)\right)$$

$$= \ln|\vec{x}|\left(1+O\left(\tfrac{1}{|\vec{x}|}\right)\right)$$

$$\Rightarrow \quad \varphi(\vec{x}) = F\,\ln|\vec{x}|\left(1+O\left(\tfrac{1}{|\vec{x}|}\right)\right)$$

with $F := \int_{\mathbb{R}^2} d^2\vec{x}'\, B(\vec{x}')$ flux $F < \infty$

- Remember: SUSY of 2D Pauli-Hamiltonian

$$H = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A\end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_-\end{pmatrix} \quad \begin{array}{l}\text{spin-up}\\ \text{spin-down}\end{array}$$

• Proof of Aharonov and Casher

only $g=+2$ (uppersign)

Assume SUSY unbroken $\Rightarrow$ $iA\psi_0^- = 0$, $iA^\dagger \psi_0^+ = 0$ with $\psi_0^\pm \in \mathcal{H}^\pm$ spin-up / spin-down

$$\Rightarrow \quad (\partial_1 + i\partial_2)\, e^{-\frac{\varphi}{\Phi_0}\mathrm{sgn}\,e}\, \psi_0^- = 0$$
$$(\partial_1 - i\partial_2)\, e^{+\frac{\varphi}{\Phi_0}\mathrm{sgn}\,e}\, \psi_0^+ = 0$$

let $z := x_1 - i x_2$ $\qquad \partial_{z^*} = \partial_1 + i\partial_2$, $\quad \partial_z = \partial_1 - i\partial_2$

$$\Rightarrow \quad f_-(z^*) := e^{-\frac{\varphi}{\Phi_0}\mathrm{sgn}\,e}\, \psi_0^- \quad \text{analytic in } z^*$$
$$f_+(z) := e^{+\frac{\varphi}{\Phi_0}\mathrm{sgn}\,e}\, \psi_0^+ \quad \text{analytic in } z$$

Now let $F > 0$ and $e < 0$ (electrons)

1) $\psi_0^+(\vec{x}) = e^{\varphi/\Phi_0} f_+(z) = f_+(z)\, |z|^{F/\Phi_0}\left(1 + O\left(\frac{1}{|z|}\right)\right) \overset{?}{\in} L^2(\mathbb{R}^2)$

$\rightsquigarrow$ $f_+(z) \to 0$ for $|z| \to \infty$ (in all directions)

and analytic (power series in z) $\Rightarrow f_+(z) \equiv 0$

$\Rightarrow$ No spin-up states with $E_0 = 0$

2) $\psi_0^-(\vec{x}) = e^{-\varphi/\Phi_0} f_-(z^*) = f_-(z^*)\, |z^*|^{-F/\Phi_0}\left(1 + O\left(\frac{1}{|z^*|}\right)\right) \overset{?}{\in} L^2(\mathbb{R})$

$\rightsquigarrow$ $f_-(z^*)$ polynomial bounded and analytic

$\rightsquigarrow$ $f_-(z^*)$ is polynomial in z^* with degree $K < \left(\frac{F}{\Phi_0} - 1\right)$

There exist exact $d := \left[\!\left[\frac{F}{\Phi_0} \right]\!\right]$ linear independent polyn.

$$\psi_0^-(\vec{x}) = \mathcal{N}\, e^{-\varphi/\Phi_0} (x_1 - i x_2)^K$$

$$K = 0, 1, 2, \ldots, d-1$$

Summary:

$$d := [\![\frac{F}{\Phi_0}]\!]$$ for electrons $e<0$

$F>0$: $\quad \psi_0^+(\vec{x}) \equiv 0$

$$\psi_{0k}^-(\vec{x}) = \mathcal{N} \exp\left\{-\frac{\varphi(\vec{x})}{\Phi_0}\right\}(x_1 - i x_2)^k$$

$k = 0, 1, 2, .., d-1$ spin-down ground states

$F<0$: $\quad \psi_0^-(\vec{x}) \equiv 0$

$$\psi_{0k}^+(x) = \mathcal{N} \exp\left\{\frac{\varphi(\vec{x})}{\Phi_0}\right\}(x_1 + i x_2)^k$$

$k = 0, 1, 2, .., d-1$ spin-up ground states

Exercise 13: Spherical sym. Pauli systems

Recall
$$H_P = \frac{1}{2m}\left(\vec{P}^2 - \frac{e}{c}\vec{A}(\vec{r})\right)^2 - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} + e\phi(\vec{r})$$

Now we consider $\vec{B}=0$ but $\vec{A}\sim\vec{\nabla}U(r)$ and $e\phi(\vec{r}) = V(r)$

↑ Tensor potential in Dirac eq.

$$\leadsto \quad H_P = \frac{1}{2m}\left(\vec{P} - i\vec{e}_r U'(r)\right)^2 + V(r) \quad \text{on } L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$$

- Conserved angular momentum ($\hbar = 1$)

Spin operator: $\vec{S} := \frac{1}{2}\vec{\sigma}$ acting on $\mathbb{C}^2$. $\leadsto \vec{S}^2 = s(s+1) = \frac{1}{2}\frac{3}{2} = \frac{3}{4}$ as $s=\frac{1}{2}$

Angular momentum: $\vec{L} = \vec{r}\times\vec{p}$ acts on $L^2(\mathbb{R}^3)$

Total angular momentum: $\vec{J} := \vec{L} + \vec{S}$

Spin orbit operator: $K := 2\vec{S}\cdot\vec{L} + 1 = \vec{\sigma}\cdot\vec{L} + 1$

- Some relations

- $\vec{J}^2 = (\vec{L}+\vec{S})^2 = \vec{L}^2 + 2\vec{S}\cdot\vec{L} + \vec{S}^2 = \vec{L}^2 + \vec{\sigma}\cdot\vec{L} + \frac{3}{4}$

$$\leadsto \vec{\sigma}\cdot\vec{L} = \vec{J}^2 - \vec{L}^2 - \frac{3}{4} \quad \Rightarrow \quad \underline{\underline{K = \vec{J}^2 - \vec{L}^2 + \frac{1}{4}}}$$

- $(\vec{\sigma}\cdot\vec{L})^2 = \sigma_i\sigma_j L_i L_j = \vec{L}^2 + i\varepsilon_{ijk}\sigma_k L_i L_j = \vec{L}^2 + \frac{i}{2}\varepsilon_{ijk}\underbrace{[L_i, L_j]}_{= i\varepsilon_{ijm}L_m}$

$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$

$$= \vec{L}^2 + \frac{i}{2}\sigma_k \underbrace{\varepsilon_{ijk}\varepsilon_{ijm}}_{=2\delta_{km}} i L_m = \vec{L}^2 - \vec{\sigma}\cdot\vec{L}$$

$$\Rightarrow \underline{\underline{\vec{L}^2 = \vec{\sigma}\cdot\vec{L}\,(\vec{\sigma}\cdot\vec{L}+1)}} \quad \text{and} \quad \underline{\underline{\vec{L}^2 = K(K-1)}}$$

• $K^2 = \vec{L}^2 + K = \vec{L}^2 + \vec{J}^2 - \vec{L}^2 + \frac{1}{4} = \underline{\underline{\vec{J}^2 + \frac{1}{4}}}$

K is a scalar quantity $\rightsquigarrow$ commutes with $\vec{J}^2$ and J_3

$\rightsquigarrow$ complete set of operators $\{\vec{J}^2, J_3, K\}$ on $L^2(S^2)\otimes\mathbb{C}^2$

with eigenvalues $\{j(j+1), m, -k\}$

$j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \qquad m = -j, \dots, +j \qquad k = ?$

From $K^2 = \vec{J}^2 + \frac{1}{4} \Rightarrow k^2 = j(j+1) + \frac{1}{4} = (j+\frac{1}{2})^2$

$\Rightarrow \quad k = \pm(j+\frac{1}{2})$ and $k = \pm 1, \pm 2, \dots \in \mathbb{Z}/\{0\}$

only sign of k is independent quantum number

(replaces $\uparrow\downarrow$)

From $\vec{L}^2 = K(K-1) \rightsquigarrow l(l+1) = (-k)(-k-1) = k(k+1)$

Hence for

$k>0$: $\quad k = l = j+\frac{1}{2}$ or $j = l - \frac{1}{2}$ $\quad \vec{L}$ and $\vec{S}$ are anti-aligned $\quad l \neq 0$

$k<0$: $\quad k = -l-1 = -(j+\frac{1}{2})$ or $j = l+\frac{1}{2}$ $\quad \vec{L}$ and $\vec{S}$ aligned

Define independent quantum number

$$s := -\operatorname{sgn} k = \pm 1 \quad \text{for } j = l \pm \frac{1}{2} \quad \begin{matrix}\text{aligned} \\ \text{anti aligned}\end{matrix}$$

Hence $\quad L^2(S^2)\otimes\mathbb{C}^2 = \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \bigoplus_{m=-j}^{+j} \bigoplus_{s=\pm 1} |jms\rangle\langle jms|$

with

$\vec{J}^2 |jms\rangle = j(j+1)|jms\rangle$

$J_3 |jms\rangle = m|jms\rangle$

$K|jms\rangle = -k|jms\rangle = s|k||jms\rangle = s(j+\frac{1}{2})|jms\rangle$

Note $\quad j = l + \frac{1}{2}s$

• Spin-spherical harmonics / Pauli-spinors

See Bjorken / Drell, "Relativistic QM"
Schwabl, "QM für Fortgeschrittene"
Thaller, "The Dirac Equation"

Let $\varphi^{(s)}_{jm}(\theta,\varphi) := \langle \theta,\varphi | jms \rangle$

$$\varphi^{(s)}_{jm}(\theta,\varphi) = \begin{pmatrix} \left[\frac{l+sm+\frac{1}{2}}{2l+1}\right]^{1/2} Y_l^{m-\frac{1}{2}}(\theta,\varphi) \\ s\left[\frac{l-sm+\frac{1}{2}}{2l+1}\right]^{1/2} Y_l^{m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix}$$

with $l = j - s/2$

$Y_l^\mu(\theta,\varphi)$ usual spherical harmonics with $Y_l^{-\mu}(\theta,\varphi) = (-1)^\mu Y_l^{\mu *}(\theta,\varphi)$

Property: $(\vec{\sigma}\cdot\vec{e}_r)\,\varphi^{(s)}_{jm}(\theta,\varphi) = \varphi^{(-s)}_{jm}(\theta,\varphi)$

changes only sign of s ❗

Proof: Recall $[L_j, x_k] = i\,\varepsilon_{jkl}\,x_l$

Consider

• $(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{L}) = \underbrace{\vec{r}\cdot\vec{L}}_{=0} + i\varepsilon_{ijk}\,x_i L_j \sigma_k = i\varepsilon_{ijk}\,\sigma_k\, x_i L_j$

• $(\vec{\sigma}\cdot\vec{L})(\vec{\sigma}\cdot\vec{r}) = \vec{L}\cdot\vec{r} + i\varepsilon_{ijk}\,\sigma_k L_i x_j \underset{i\leftrightarrow j}{=} -i\varepsilon_{ijk}\,\sigma_k L_j x_i$

$$\Rightarrow \{(\vec{\sigma}\cdot\vec{r}), (\vec{\sigma}\cdot\vec{L})\} = i\varepsilon_{ijk}\,\sigma_k\underbrace{(x_i L_j - L_j x_i)}_{= i\,\varepsilon_{ijl}\,x_l} = -\underbrace{\varepsilon_{ijk}\,\varepsilon_{ijl}}_{2\delta_{kl}}\,\sigma_k x_l$$

$$= -2\,\vec{\sigma}\cdot\vec{r}$$

Hence

$$\{K, \vec{\sigma}\cdot\vec{e}_r\} = (\vec{\sigma}\cdot\vec{L}+1)(\vec{\sigma}\cdot\vec{e}_r) + (\vec{\sigma}\cdot\vec{e}_r)(\vec{\sigma}\cdot\vec{L}+1)$$

$$= \{(\vec{\sigma}\cdot\vec{L}), (\vec{\sigma}\cdot\vec{e}_r)\} + 2(\vec{\sigma}\cdot\vec{e}_r) = \frac{1}{r}\Big[\underbrace{\{(\vec{\sigma}\cdot\vec{L}), (\vec{\sigma}\cdot\vec{r})\}}_{=-2\vec{\sigma}\cdot\vec{r}} + 2(\vec{\sigma}\cdot\vec{r})\Big] = 0$$

$$\Rightarrow \{K, \vec{\sigma}\cdot\vec{e}_r\} = 0$$

Let $K|\kappa\rangle = -\kappa|\kappa\rangle$ be eigenstate of K

$\rightsquigarrow$ $K(\vec{\sigma}\cdot\vec{e}_r)|\kappa\rangle = -(\vec{\sigma}\cdot\vec{e}_r)K|\kappa\rangle = +\kappa(\vec{\sigma}\cdot\vec{e}_r)|\kappa\rangle$

$\rightsquigarrow$ $(\vec{\sigma}\cdot\vec{e}_r)|\kappa\rangle$ is eigenstate of K with eigenvalue $+\kappa$

$$\Rightarrow (\vec{\sigma}\cdot\vec{e}_r)\,\varphi^{(s)}_{jm} = \varphi^{(-s)}_{jm}$$

- Grading of $\mathcal{H} = L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$

Let $\mathcal{H}^\pm := \mathcal{H}|_{s=\pm1}$ subspace where $\vec{S}$ and $\vec{L}$ are parallel / anti-parallel

subspaces where $l = j \mp \frac{1}{2}$

Define a Witten operator

$$W := -\frac{K}{|K|} \qquad \text{well defined as } K \neq 0$$

In matrix notation

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{array}{l} s=+1 \\ s=-1 \end{array}$$

Obviously $(\vec{\sigma}\cdot\vec{e}_r) : \mathcal{H}^\pm \to \mathcal{H}^\mp$

may generate SUSY transformations

Exercise 14: Some examples

• The free Pauli Hamiltonian

Homework: $\vec{\sigma}\cdot\vec{p} = -i\hbar\,(\vec{\sigma}\cdot\vec{e}_r)\left(\partial_r - \frac{K-1}{r}\right)$

$$\rightsquigarrow\ \vec{\sigma}\cdot\vec{p} : \mathcal{H}^{\pm} \to \mathcal{H}^{\mp}$$

Supercharge: $Q = \frac{1}{\sqrt{2m}}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ with $A := \vec{\sigma}\cdot\vec{p} = A^{+}$

$\rightsquigarrow$ Hamiltonian: $H := \{Q, Q^{+}\} = \frac{\vec{p}^{\,2}}{2m}$

• General spherically symmetric potentials

$$H = \frac{1}{2m}\vec{p}^{\,2} + V(r) = \frac{1}{2m}\left(p_r^2 + \frac{\vec{L}^2}{r^2}\right) + V(r)$$

$p_r := -i\hbar\left(\partial_r + \frac{1}{r}\right) = -i\hbar\,\frac{1}{r}\,\partial_r\, r$ radial momentum op.

Let us restrict $\mathcal{H}$ to subspace with fixed j and m

$\rightsquigarrow\ \mathcal{H}|_{jm} = \mathcal{H}^{\pm}$ with $s = \pm 1$

recall $\ell^{(s)} = j - \frac{1}{2}s$ $\rightsquigarrow$ and set $\ell^{(-)} \equiv \ell$ $\rightsquigarrow$ $\ell^{(+)} = \ell - 1$

and $K^2 = (j+\frac{1}{2})^2 \rightsquigarrow K = \begin{cases} -(j+\frac{1}{2}) = -(\ell^{(+)}+1) = -\ell & \text{on } \mathcal{H}^{+} \\ (j+\frac{1}{2}) = \ell^{(-)} = \ell & \text{on } \mathcal{H}^{-} \end{cases}$

Hence $H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$ with $\vec{L}^2 = K(K-1)\ (= K(K+1))$

$$H_{\pm} = \frac{p_r^2}{2m} + \frac{\hbar^2\,\ell(\ell \mp 1)}{2mr^2} + V(r)$$

see Witten models on half line!

- **The Coulomb problem** $V(r) = -\frac{\alpha}{r}$

without any proofs, for info only

Runge-Lenz vector : $\vec{R} := \frac{1}{2\alpha m}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - \vec{e}_r$ is conserved

$$[H,\vec{R}] = \vec{0}\ ,\quad \vec{R}\cdot\vec{L} = 0\ ,\quad [L_j, R_k] = i\varepsilon_{jkl} R_l$$

$$|\vec{R}|^2 = \frac{2H}{m\alpha^2}(\vec{L}^2+1)+1$$

$$[R_i, R_j] = -\frac{2iH}{m\alpha^2}\varepsilon_{ijk} L_k\ ,\quad (\vec{\sigma}\cdot\vec{R})^2 = 1 + \frac{2HK^2}{m\alpha^2}$$

$$\vec{\sigma}\cdot\vec{R} = (\vec{\sigma}\cdot\vec{e}_r)\left[-\frac{K}{m\alpha}\left(\partial_r - \frac{K-1}{r}\right) - 1\right]$$

$\uparrow$ generates SUSY transformations between $\mathcal{H}^\pm$

Let $Q_1 := \sqrt{\frac{m\alpha^2}{4K^2}}(\vec{\sigma}\cdot\vec{R}) \rightsquigarrow H_s = 2Q_1^2 = H + \frac{m\alpha^2}{2K^2}$

with fixed j and m

$$H_s\big|_{jm} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \frac{p_r^2}{2m} + \frac{\hbar^2 l(l\mp 1)}{2mr^2} - \frac{\alpha}{r} + \frac{m\alpha^2}{2l^2}$$

- **Spherical tensor potentials**

Let $A := \frac{1}{\sqrt{2m}}\vec{\sigma}\cdot(\vec{p} - i\vec{e}_r U'(r))$, $U : \mathbb{R}^+ \to \mathbb{R}$

$$\rightsquigarrow H_s := \{Q, Q^\dagger\} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} \text{ with}$$

$$H_\pm = \frac{\vec{p}^2}{2m} + \frac{1}{2m}[U'(r)]^2 \pm \frac{1}{2m}U''(r) \pm \frac{U'(r)}{mr}K$$

Example: $U(r) = \frac{m}{2}\omega^2 r^2$ Pauli oscillator

$$\rightsquigarrow H_\pm = \frac{\vec{p}^2}{2m} + \frac{m}{2}\omega^2\vec{r}^2 \pm \omega(K+\tfrac{1}{2})$$