Exercise 8 Details for N=1 Case in Darboux Method

We have
$$H_\pm = -\frac{\hbar^2}{2m}\partial_x^2 + V_\pm(x)$$
$$A = \frac{\hbar}{\sqrt{2m}}\partial_x + \Phi(x)$$

Condition: $H_+ A = A H_-$

LHS:
$$H_+ A = \left(-\frac{\hbar^2}{2m}\partial_x^2 + V_+\right)\left(\frac{\hbar}{\sqrt{2m}}\partial_x + \Phi\right)$$
$$= -\left(\frac{\hbar}{\sqrt{2m}}\right)^3\partial_x^3 - \frac{\hbar^2}{2m}\partial_x^2\Phi + V_+\frac{\hbar}{\sqrt{2m}}\partial_x + V_+\Phi$$
$$\partial_x^2\Phi = \Phi'' + 2\Phi'\partial_x + \Phi\partial_x^2$$
$$= -\left(\frac{\hbar}{\sqrt{2m}}\right)^3\partial_x^3 - \frac{\hbar^2}{2m}\left[\Phi'' + 2\Phi'\partial_x + \Phi\partial_x^2\right] + \frac{\hbar}{\sqrt{2m}}V_+\partial_x + V_+\Phi$$
$$= \underset{③}{-\left(\frac{\hbar}{\sqrt{2m}}\right)^3\partial_x^3} \underset{②}{- \frac{\hbar^2}{2m}\Phi\partial_x^2} + \underbrace{\left(\frac{\hbar}{\sqrt{2m}}V_+ - \frac{2\hbar^2}{2m}\Phi'\right)}_{①}\partial_x + \underbrace{\left(V_+\Phi - \frac{\hbar^2}{2m}\Phi''\right)}_{⓪}$$

RHS:
$$A H_- = \left(\frac{\hbar}{\sqrt{2m}}\partial_x + \Phi\right)\left(-\frac{\hbar^2}{2m}\partial_x^2 + V_-\right)$$
$$= -\left(\frac{\hbar}{\sqrt{2m}}\right)^3\partial_x^3 - \frac{\hbar^2}{2m}\Phi\partial_x^2 + \frac{\hbar}{\sqrt{2m}}\left(V_-' + V_-\partial_x\right) + \Phi V_-$$
$$= \underset{③}{-\left(\frac{\hbar}{\sqrt{2m}}\right)^3\partial_x^3} \underset{②}{- \frac{\hbar^2}{2m}\Phi\partial_x^2} + \underset{①'}{\frac{\hbar}{\sqrt{2m}}V_-\partial_x} + \underset{⓪'}{\left(\frac{\hbar}{\sqrt{2m}}V_-' + \Phi V_-\right)}$$

① = ①' :
$$\frac{\hbar}{\sqrt{2m}}V_- = \frac{\hbar}{\sqrt{2m}}V_+ - \frac{2\hbar^2}{2m}\Phi'$$
$$\boxed{V_-(x) = V_+(x) - \frac{2\hbar}{\sqrt{2m}}\Phi'(x)} \quad \text{(I)} \qquad \text{(cf. Witten model)}$$

⓪ = ⓪' :
$$\boxed{\frac{\hbar}{\sqrt{2m}}V_-' + \Phi V_- = \Phi V_+ - \frac{\hbar^2}{2m}\Phi''} \quad \text{(II)}$$

(I) in (II):
$$\frac{\hbar}{\sqrt{2m}}\left(V_+' - \cancel{\frac{2\hbar}{\sqrt{2m}}\Phi''}^{\,1}\right) + \Phi\left(\cancel{V_+}^{\,2} - \frac{2\hbar}{\sqrt{2m}}\Phi'\right) = \cancel{\Phi V_+}^{\,2} - \cancel{\frac{\hbar^2}{2m}\Phi''}^{\,1}$$
$$V_+' - \frac{\hbar}{\sqrt{2m}}\Phi'' - 2\Phi\Phi' = 0$$

Integration:

$$V_+(x) - \frac{\hbar}{\sqrt{2m}}\Phi'(x) - \Phi^2(x) = \varepsilon \qquad \text{const.} \quad \otimes$$

$$\underline{\underline{V_+(x) = \Phi^2(x) + \frac{\hbar}{\sqrt{2m}}\Phi'(x) + \varepsilon}} \qquad \text{Riccati equ.}$$

$$\underline{\underline{V_-(x) = \Phi^2(x) - \frac{\hbar}{\sqrt{2m}}\Phi'(x) + \varepsilon}}$$

Ansatz: $\Phi(x) = \frac{\hbar}{\sqrt{2m}}\frac{u'}{u} \curvearrowright \Phi'(x) = \frac{\hbar}{\sqrt{2m}}\frac{uu''-u'^2}{u^2}$ in $\otimes$

$$V_+ - \frac{\hbar^2}{2m}\left(\frac{u''}{u} - \frac{u'^2}{u^2}\right) - \frac{\hbar^2}{2m}\left(\frac{u'}{u}\right)^2 = \varepsilon$$

$$-\frac{\hbar^2}{2m}\frac{u''}{u} + V_+ = \varepsilon \implies \left[-\frac{\hbar^2}{2m}\partial_x^2 + V_+(x)\right]u(x) = \varepsilon\, u(x)$$

shifted Witten model:

$$H_+ = AA^+ + \varepsilon$$
$$H_- = A^+A + \varepsilon$$

From Riccati eq.: $-\frac{\hbar}{\sqrt{2m}}\Phi' = \Phi^2 + \varepsilon - V_+$

$$\curvearrowright \quad V_- = \Phi^2 + \Phi^2 + \varepsilon - V_+ + \varepsilon = 2\Phi^2 + 2\varepsilon - V_+$$

$$\underline{\underline{V_-(x) = \frac{\hbar^2}{m}\left(\frac{u'(x)}{u(x)}\right)^2 - V_+(x) + 2\varepsilon}}$$

Norm.: $|\phi_n^-\rangle = C_n A^+|\phi_n^+\rangle \curvearrowright \|\phi_n^-\|^2 = |C_n|^2\langle\phi_n^+|AA^+|\phi_n^+\rangle$

$$= |C_n|^2\langle\phi_n^+|H_+ - \varepsilon|\phi_n^+\rangle = |C_n|^2(E_n - \varepsilon) > 0$$

as $\varepsilon < E_0$

Hence

$$\phi_n^-(x) = \frac{1}{\sqrt{E_n-\varepsilon}}A^+\phi_n^+(x) = \frac{1}{\sqrt{E_n-\varepsilon}}\left(-\frac{\hbar}{\sqrt{2m}}\partial_x + \frac{\hbar}{\sqrt{2m}}\frac{u'}{u}\right)\phi_n^+(x)$$

Exercise 9: Generalised Creation and Annihilation Operators

HO algebra:

$$a = \frac{1}{\sqrt{2}}(\partial_x + x) \qquad a^+ = \frac{1}{\sqrt{2}}(-\partial_x + x)$$

$$H_+ = a^+ a + \frac{1}{2} = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$$

$$[H_+, a] = -a \quad , \quad [H_+, a^+] = a^+ \quad , \quad [a, a^+] = 1$$

with

$$a|\phi_n^+\rangle = \sqrt{n}\,|\phi_{n-1}^+\rangle \qquad a^+|\phi_n^+\rangle = \sqrt{n+1}\,|\phi_{n+1}^+\rangle$$

$$H_+|\phi_n^+\rangle = E_n|\phi_n^+\rangle \qquad E_n = n + \frac{1}{2}$$

SUSY Partner:

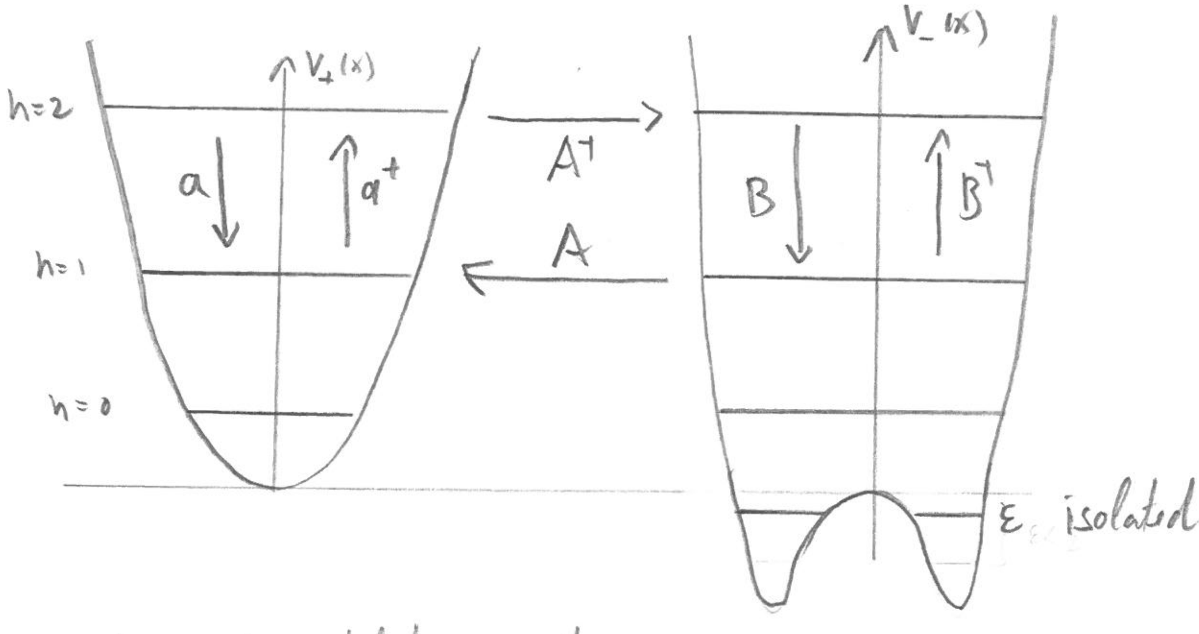
$$H_- = -\frac{1}{2}\partial_x^2 + \left(\frac{u'}{u}\right)^2 - \frac{x^2}{2} + 2\varepsilon$$

$$\phi_\varepsilon^-(x) = \frac{c}{u(x)} \qquad H_-|\phi_\varepsilon^-\rangle = \varepsilon|\phi_\varepsilon^-\rangle$$

$$|\phi_n^-\rangle = \frac{1}{\sqrt{E_n - \varepsilon}} A^+|\phi_n^+\rangle$$

$$|\phi_n^+\rangle = \frac{1}{\sqrt{E_n - \varepsilon}} A|\phi_n^-\rangle \qquad A = \frac{1}{\sqrt{2}}\left(\partial_x + \frac{u'}{u}\right)$$

$$H|\phi_n^-\rangle = E_n|\phi_n^-\rangle$$



New creation and annihilation operators:

$$B^+ := A^+ a^+ A \qquad B = A^+ a A$$

The B-Algebra:

$$B|\phi^-_{n+1}\rangle = A^+ a A |\phi^-_{n+1}\rangle$$
$$= (E_{n+1}-\varepsilon)^{1/2} A^+ a |\phi^+_{n+1}\rangle$$
$$= [(E_{n+1}-\varepsilon)(n+1)]^{1/2} A^+ |\phi^+_n\rangle$$
$$= [(E_{n+1}-\varepsilon)(n+1)(E_n-\varepsilon)]^{1/2} |\phi^-_n\rangle \qquad \text{annihilation}$$

$$B^+|\phi^-_n\rangle = A^+ a^+ A |\phi^-_n\rangle$$
$$= (E_n-\varepsilon)^{1/2} A^+ a^+ |\phi^-_n\rangle$$
$$= [(E_n-\varepsilon)(n+1)]^{1/2} A^+ |\phi^-_{n+1}\rangle$$
$$= [(E_n-\varepsilon)(n+1)(E_{n+1}-\varepsilon)]^{1/2} |\phi^-_{n+1}\rangle \qquad \text{creation}$$

$$\Rightarrow \quad H_- B|\phi^-_n\rangle = E_{n-1} B|\phi^-_n\rangle$$
$$B H_- |\phi^-_n\rangle = E_n B|\phi^-_n\rangle$$

$$\rightsquigarrow \quad [H_-, B]|\phi^-_n\rangle = (E_{n-1}-E_n) B|\phi^-_n\rangle = -B|\phi^-_n\rangle$$

$$H_- B^+|\phi^-_n\rangle = E_{n+1} B^+|\phi^-_n\rangle$$
$$B^+ H_- |\phi^-_n\rangle = E_n B^+|\phi^-_n\rangle$$

$$\rightsquigarrow \quad [H_-, B^+]|\phi^-_n\rangle = (E_{n+1}-E_n) B^+|\phi^-_n\rangle = +B^+|\phi^-_n\rangle$$

Hence $\quad [H_-, B] = -B \qquad [H_-, B^+] = +B^+$

holds also on zero mode space spanned by $|\phi^-_\varepsilon\rangle$

$$BB^+|\phi^-_n\rangle = (E_{n+1}-\varepsilon)(n+1)(E_n-\varepsilon)|\phi^-_n\rangle = (E_n+1-\varepsilon)(E_n+\tfrac{1}{2})(E_n-\varepsilon)|\phi^-_n\rangle$$

$$B^+B|\phi^-_n\rangle = (E_n-\varepsilon)\, n\,(E_{n-1}-\varepsilon)|\phi^-_n\rangle = (E_n-\varepsilon)(E_n-\tfrac{1}{2})(E_n+1-\varepsilon)|\phi^-_n\rangle$$

$$\Rightarrow \quad BB^+ = (H_-+1-\varepsilon)(H_-+\tfrac{1}{2})(H_--\varepsilon)$$

$$B^+B = (H_--\varepsilon)(H_--\tfrac{1}{2})(H_--1-\varepsilon)$$

Hence $[B, B^+] = (H_- - \varepsilon)\left[(H_- + 1 - \varepsilon)(H_- + \tfrac{1}{2}) - (H_- - \tfrac{1}{2})(H_- - 1 - \varepsilon)\right]$

$$= (H_- - \varepsilon)\left[H_-^2 + \tfrac{1}{2}H_- + H_- + \tfrac{1}{2} - \varepsilon H_- - \tfrac{\varepsilon}{2} - \left(H_-^2 - H_- - \varepsilon H_- - \tfrac{1}{2}H_- + \tfrac{1}{2} + \tfrac{\varepsilon}{2}\right)\right]$$

$$= (H_- - \varepsilon)(3H_- - \varepsilon) = 3H_-^2 - \varepsilon H_- - 3\varepsilon H_- + \varepsilon^2$$

$$= 3H_-^2 - 4\varepsilon H_- + \varepsilon^2$$

holds also on span(Φ_ε)

Non-linear (quadratic) algebra:

$$[H_-, B] = -B\,, \quad [H_-, B^+] = B^+\,, \quad [B, B^+] = 3H_-^2 - 4\varepsilon H_- + \varepsilon^2$$

Remarks: For the radial HO one can construct analogous raising and lowering operators. The corresponding B-algebra becomes a cubic algebra. Here both, broken and unbroken SUSY, can be realized

↗ GJ and P. Roy

Exercise 10: Fluctuation of energy functional

Fluctuation about static solution: $\phi = \phi_{st} + \psi$, ψ small, $\psi \to 0$ for $x \to \pm\infty$

$$\rightsquigarrow \quad E[\phi] = \int dx \left[\tfrac{1}{2} (\phi'_{st} + \psi')^2 + U(\phi_{st} + \psi) \right]$$

$$\approx \int dx \left[\tfrac{1}{2} (\phi'_{st} + \psi')^2 + U(\phi_{st}) + U'(\phi_{st}) \psi + \tfrac{1}{2} U''(\phi_{st}) \psi^2 \right]$$

$$= \underbrace{\int dx \left[\tfrac{1}{2} \phi'^{2}_{st} + U(\phi_{st}) \right]}_{= E[\phi_{st}]} + \int dx \left[\phi'_{st} \psi' + U'(\phi_{st}) \psi \right] + \int dx \left[\tfrac{1}{2} \psi'^2 + \tfrac{1}{2} U''(\phi_{st}) \psi^2 \right]$$

$$\int dx\, \phi'_{st} \psi' = \phi'_{st} \psi \Big|_{-\infty}^{+\infty} - \int dx\, \phi''_{st} \psi = - \int dx\, \phi''_{st} \psi$$

$$\int dx\, \psi'^2 = \psi \psi' \Big|_{-\infty}^{+\infty} - \int dx\, \psi \psi'' = - \int dx\, \psi\, \partial_x^2 \psi$$

$$= E[\phi_{st}] + \int dx \big[\underbrace{- \phi''_{st} + U'(\phi_{st})}_{=0} \big] \psi + \underbrace{\int dx\, \tfrac{1}{2} \psi \left[- \partial_x^2 + U''(\phi_{st}) \right] \psi}_{= \delta E[\psi]}$$

$$\rightsquigarrow \quad \delta E[\psi] = \frac{1}{2} \int dx\, \psi(x) \underbrace{\left(- \partial_x^2 + U''(\phi_{st}(x)) \right)}_{= H \text{ fluctuation operator}} \psi(x)$$

Exercise 11: Further Examples of Field Models via SUSY

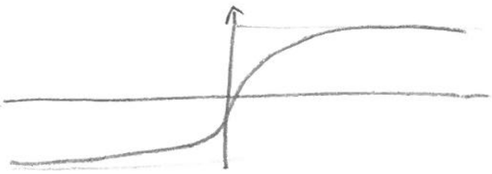
a) The ϕ^4-Theory:

Choose $W(x) = 2\tanh x$

$$\rightsquigarrow \psi_0(x) = N \exp\left\{-\int dx\, 2\tanh x\right\} = \frac{N}{\cosh^2 x}$$

$$\phi_{st}(x) = C \int dx\, \psi_0(x) = \tanh x \quad , \quad C = 1/N$$

$$\rightsquigarrow \phi_\pm = \pm 1$$



$$U(\phi_{st}) = \frac{1}{2}\left(\phi_{st}'(x)\right)^2 = \frac{1}{2\cosh^2 x} = \frac{1}{2}\left(1-\tanh^2 x\right)^2 = \frac{1}{2}\left(1-\phi_{st}^2\right)^2$$

Hence $\underline{\underline{U(\phi) = \frac{1}{2}\left(1-\phi^2\right)^2}}$

b) A new Field Model

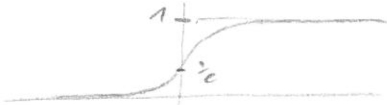
Choose $W(x) = 1 - e^{-x}$ $\quad \rightsquigarrow V_-(x) = W^2(x) - W'(x) = e^{-2x} - 3e^{-x} + 1$

Morse potential

$$\rightsquigarrow \psi_0(x) = N \exp\{-x - e^{-x}\}$$

$$\phi_{st}(x) = \exp\{-e^{-x}\} \quad \left(\rightsquigarrow \phi_{st}' = \exp\{-e^{-x}\}\, e^{-x} = \psi_0(x) \quad N=1\right)$$

$$\rightsquigarrow \phi_+ = 1 \quad , \quad \phi_- = 0$$



$$U(\phi_{st}) = \frac{1}{2}\phi_{st}'^2(x) = \frac{1}{2}\left(e^{-x}\exp\{-e^{-x}\}\right)^2 = \frac{1}{2}\left(\ln\phi_{st}\,\exp(\ln\phi_{st})\right)^2$$

$$e^{-x} = -\ln\phi_{st}$$

$$= \frac{1}{2}\phi_{st}^2 \ln^2\phi_{st}$$

$$\rightsquigarrow \underline{\underline{U(\phi) = \frac{1}{2}\phi^2 \ln^2\phi}}$$