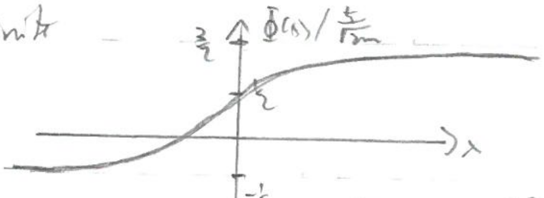
Exercise 5: SUSY Transformation for Continuous States

Witten model: $H_\pm = \frac{p^2}{2m} + V_\pm(x)$

$$V_\pm(x) = \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}} \Phi'(x)$$

Assume: $\lim_{x\to-\infty} V_\pm(x) = |\Phi_-|^2 \qquad \lim_{x\to+\infty} V_\pm(x) = |\Phi_+|^2$

with $|\Phi_-|^2 \leq |\Phi_+|^2$ finite



Example: $\Phi(x) = \frac{\hbar}{\sqrt{2m}}\left(\tanh x + \frac{1}{2}\right)$

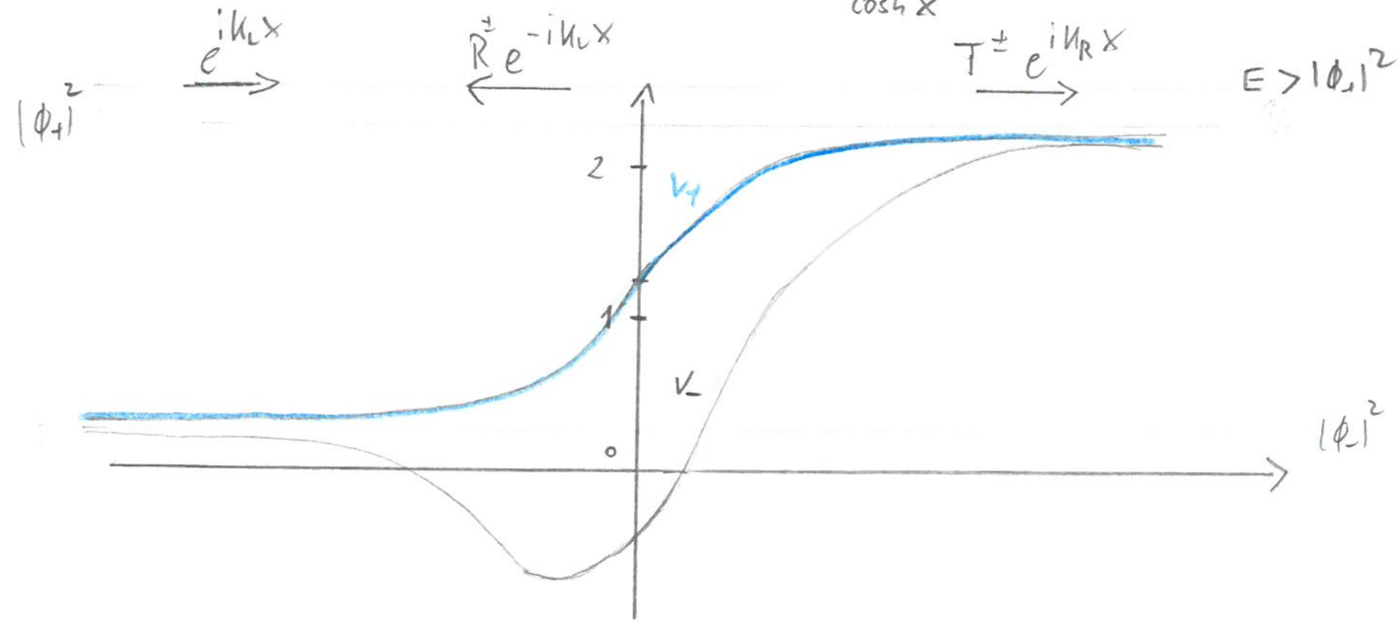
SUSY good?

$$\leadsto |\Phi_-|^2 = \frac{1}{4}\frac{\hbar^2}{2m}, \quad |\Phi_+|^2 = \frac{9}{4}\frac{\hbar^2}{2m}$$

$$V_\pm(x) = \frac{\hbar^2}{2m}\left((\tanh x + \tfrac{1}{2})^2 \pm \frac{1}{\cosh^2 x}\right) = \frac{\hbar^2}{2m}\left(\tanh x + \frac{1}{4} + \frac{\sinh^2 x \pm \cosh^2 x}{\cosh^2 x}\right)$$

$$\phi_0^-(x) = N \exp\{-\int dx\,(\tanh x + \tfrac{1}{2})\} = N\frac{e^{-x/2}}{\cosh x}$$

only 1 bound state?



SUSY Transformations: $|\phi_E^-\rangle = N A^\dagger |\phi_E^+\rangle \qquad |\phi_E^+\rangle = N A |\phi_E^-\rangle \qquad E > 0$

$\hookrightarrow N = \frac{1}{\sqrt{E}}$ for L^2 functions?

$\leadsto$ x-representation $\qquad A = \frac{\hbar}{\sqrt{2m}}\partial_x + \phi(x)$

$$\underline{\left[\pm\frac{\hbar}{\sqrt{2m}}\partial_x + \Phi(x)\right]\phi_E^\mp(x) = \frac{1}{N}\,\phi_E^\pm(x)}$$

Incoming plane wave from left with $E > |\Phi_+|^2$ $\rightsquigarrow$ transmission and refl.

wave numbers: $k_L(E) := \frac{1}{\hbar}\left[2m(E - \Phi_-^2)\right]^{1/2}$

$$k_R(E) := \frac{1}{\hbar}\left[2m(E - \Phi_+^2)\right]^{1/2}$$

Asymp. wave functions:

$$x \to -\infty : \quad \phi_E^{\pm}(x) \sim e^{ik_L x} + R^{\pm} e^{-ik_L x}$$

$$x \to +\infty : \quad \phi_E^{\pm}(x) \sim T^{\pm} e^{ik_R x}$$

$R^{\pm}$: reflection amplitude for $V_{\pm}$ $\qquad |R^{\pm}|^2$ = refl. probability

$T^{\pm}$: transmission amplitude for $V_{\pm}$ $\qquad |T^{\pm}|^2$ = trans. prob.

- Consider SUSY Transformation for $x \to \pm\infty$ $\qquad (A\phi^- = C\phi^+$

1. $x \to -\infty$ for $\phi_E(x)$: $\left(\frac{\hbar}{\sqrt{2m}}\partial_x + \Phi_-\right)\left(e^{ik_L x} + R^- e^{-ik_L x}\right) = C\left(e^{ik_L x} + R^+ e^{-ik_L x}\right)$

$$\rightsquigarrow \left(\frac{i\hbar}{\sqrt{2m}}k_L + \Phi_-\right)e^{ik_L x} + R^-\left(-\frac{i\hbar}{\sqrt{2m}}k_L + \Phi_-\right)e^{-ik_L x} = Ce^{ik_L x} + \frac{1}{N}R^+ e^{-ik_L x}$$

$$\Rightarrow C = \frac{i\hbar}{\sqrt{2m}}k_L + \Phi_- \quad , \quad CR^+ = \left(\Phi_- - \frac{i\hbar}{\sqrt{2m}}k_L\right)R^-$$

$$\Rightarrow \quad \boxed{R^+(E) = \frac{\Phi_- - i\sqrt{E - \Phi_-^2}}{\Phi_- + i\sqrt{E - \Phi_-^2}}\, R^-(E)}$$

In particular: $|R^+(E)|^2 = |R^-(E)|^2$

partner potentials have same reflexion prob.

$R^+(E) = 0 \iff R^-(E) = 0$

reflexionlessness is simultaneous property of $V_{\pm}$

2) $x \to +\infty$ for $\phi_E^-(x)$: $\left(\frac{\hbar}{\sqrt{2m}}\partial_x + \Phi_+\right) T^- e^{ik_R x} = C\, T^+ e^{ik_R x}$

$$\rightsquigarrow \left(\frac{i\hbar}{\sqrt{2m}} k_R + \Phi_+\right) T^- = \left(\frac{i\hbar}{\sqrt{2m}} k_L + \Phi_-\right) T^+$$

$$\Rightarrow \quad \boxed{T^+(E) = \frac{\Phi_+ + i\sqrt{E - \Phi_+^2}}{\Phi_- + i\sqrt{E - \Phi_-^2}}\, T^-(E)}$$

In particular: $|T^+(E)|^2 = \frac{\Phi_+^2 + E - \Phi_+^2}{\Phi_-^2 + E - \Phi_-^2} |T^-(E)|^2 = |T^-(E)|^2$

Exercise 6: Further Examples of Witten Models

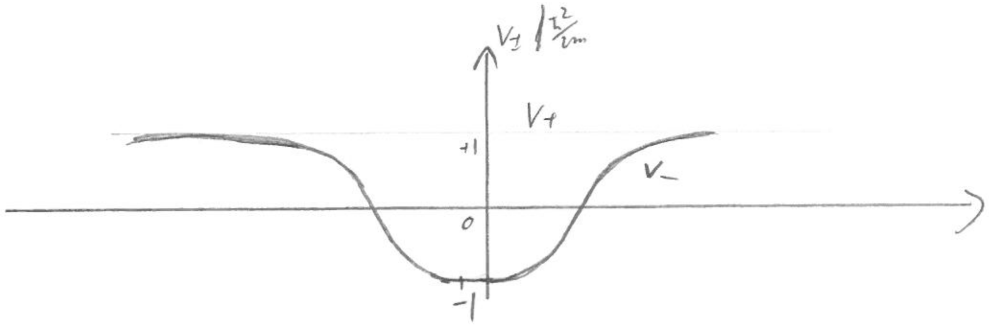
a) A SUSY Partner of the free particle on $\mathbb{R}$

Consider $\Phi(x) := \frac{\hbar}{\sqrt{2m}} \tanh x$

$\rightsquigarrow \Phi^2(x) = \frac{\hbar^2}{2m} \tanh^2 x$, $\frac{\hbar}{\sqrt{2m}} \Phi'(x) = \frac{\hbar^2}{2m} \frac{1}{\cosh^2 x} = \frac{\hbar^2}{2m}(1 - \tanh^2 x)$

$\Rightarrow V_+(x) = \frac{\hbar^2}{2m} = \text{const}$ free particle

$V_-(x) = \frac{\hbar^2}{2m}\left(1 - \frac{2}{\cosh^2 x}\right)$ mod. Pöschel-Teller



Note: $V(x) = -\frac{\hbar^2}{2m} \frac{2}{\cosh^2 x}$ is reflexionless !

Shape invariance : $V_\pm(n,x) := \frac{\hbar^2}{2m}\left[n^2 - \frac{n(n\mp 1)}{\cosh^2 x}\right]$ with $n=1$ $V(x) = V_-(1,x)$

$\rightsquigarrow V_+(n+1,x) = V_-(n,x) + \text{const}$

$\Rightarrow$ all $V(x) = -\frac{\hbar^2}{2m} \frac{n(n+1)}{\cosh^2 x}$ $n \in \mathbb{N}$ are reflexionless !

b) The radial Harmonic Oscillator

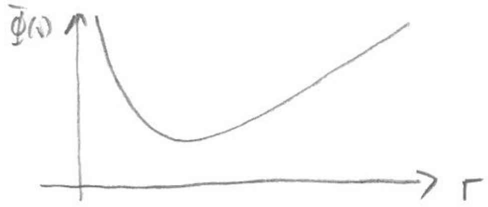
$\mathcal{H}_\pm = L^2(\mathbb{R}^+)$ with $\phi^\pm_E(0) = 0$ Dirichlet Condition at $r=0$

SUSY potential: $\Phi(r) := \sqrt{\frac{m}{2}}\,\omega r - \frac{\hbar}{\sqrt{2m}}\,\frac{\eta}{r}$, $\omega > 0$

- $\eta \geq 0$: SUSY unbroken

$$\phi_0^-(r) = \mathcal{N}\, r^\eta\, e^{-\frac{m\omega}{2\hbar} r^2}$$

- $\eta < 0$: SUSY broken



- Partner Potentials

$$V_\pm(r) = \frac{m}{2}\omega^2 r^2 + \frac{\hbar^2}{2m}\,\frac{\eta(\eta\pm 1)}{r^2} - \hbar\omega\left(\eta \mp \frac{1}{2}\right)$$

$\eta = l+1$, $l = 0,1,2,\ldots$ radial harmonic oscillator with good SUSY

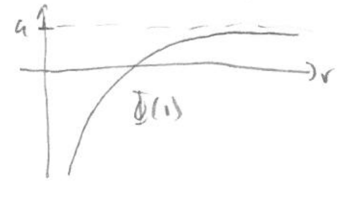
as $\eta(\eta\pm1) = (l+1)(l+1\pm1) = \begin{cases}(l+1)(l+2)\\ l(l+1)\end{cases}$

$\eta = -l$, $l = 1,2,\ldots$ radial harmonic oscillator with broken SUSY

as $\eta(\eta\pm1) = -l(-l\pm1) = l(l\mp1)$

Homework: Radial Hydrogen Atom

$\Phi(r) = a - \frac{b}{r}$ $\quad$ $a,b>0 \rightsquigarrow$ good SUSY



c) Particle in a Box and its SUSY partner

$\mathcal{H}_\pm = L^2([-\frac{\pi}{2}, \frac{\pi}{2}])$ with $\phi_E^\pm(\pm\frac{\pi}{2}) = 0$ Dirichlet

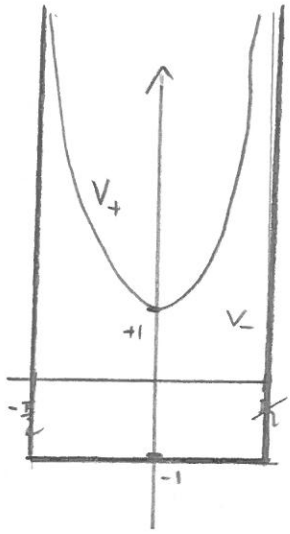
SUSY potential: $\Phi(x) := \frac{\hbar}{\sqrt{2m}} \tan x$ SUSY good

$\rightsquigarrow \Phi^2(x) = \frac{\hbar^2}{2m} \tan^2 x \ , \ \frac{\hbar}{\sqrt{2m}} \phi'(x) = + \frac{\hbar^2}{2m} \frac{1}{\cos^2 x} = + \frac{\hbar^2}{2m}(1 + \tan^2 x)$

$\Rightarrow V_+(x) = \frac{\hbar^2}{2m}\left(\frac{2}{\cos^2 x} - 1\right)$ Pöschl-Teller

$V_-(x) = -\frac{\hbar^2}{2m}$ Particle in Box

$\phi_0^-(x) = \sqrt{\frac{2}{\pi}} \cos x$



Exercise 7: On the Quasi-classical Approximation for the radial HO

Known: $H = \frac{p^2}{2m} + \underbrace{\frac{m}{2}\omega^2 r^2 + \frac{\hbar^2 l(l+1)}{2mr^2}}_{=: V_{eff}(r) = \frac{m}{2}\omega^2 r^2 + \frac{\hbar^2\lambda^2}{2mr^2}}$ $\qquad E_{nl} = \hbar\omega(2n + l + \frac{3}{2})$ $\quad n \in \mathbb{N}_0$, $l \in \mathbb{N}_0$

$\lambda^2 := l(l+1)$

WKB-Formula:

$$\int_{r_L}^{r_R} dr \left[2m\left(E - \frac{m}{2}\omega^2 r^2 - \frac{\hbar^2}{2m}\frac{\lambda^2}{r^2}\right)\right]^{1/2} = \hbar\pi(n + \tfrac{1}{2})$$

Consider $I := \int_{r_L}^{r_R} dr \left[2mE - m^2\omega^2 r^2 - \frac{\hbar^2\lambda^2}{r^2}\right]^{1/2} = \frac{1}{2}\int_{x_L}^{x_R} dx \frac{1}{x}\left[2mEx - m^2\omega^2 x^2 - \hbar^2\lambda^2\right]^{1/2}$

$x = r^2$
$dr = \frac{dx}{2x}\sqrt{x}$

$$= \frac{m\omega}{2}\int_{x_L}^{x_R}\frac{dx}{x}\left(bx - x^2 - c\right)^{1/2} \quad \text{with } b := \frac{2E}{m\omega^2} \quad c := \frac{\hbar^2\lambda^2}{m^2\omega^2}$$

$$x_{L/R} = -\tfrac{1}{2}\left(-b \pm \sqrt{b^2 - 4c}\right) = \frac{b}{2} \mp \sqrt{b^2/4 - c}$$

$$I = \frac{m\omega}{2}\underbrace{\int_{x_L}^{x_R} dx \frac{1}{x}\sqrt{(x_R - x)(x - x_L)}}_{= \frac{\pi}{2}(x_R + x_L) - \pi\sqrt{x_R x_L}}$$

in case $\lambda < 0$?

but here $\lambda > 0$

$$= \frac{m\omega}{2}\left[\frac{\pi}{2}b - \pi\sqrt{c}\right] = \frac{m\omega\pi}{2}\left[\frac{E}{m\omega^2} - \frac{\hbar|\lambda|}{m\omega}\right] = \frac{\pi\hbar}{2}\left[\frac{E}{\hbar\omega} - \lambda\right]$$

$$I \overset{!}{=} \hbar\pi(n + \tfrac{1}{2}) \rightsquigarrow \tfrac{1}{2}\left[\frac{E}{\hbar\omega} - \lambda\right] = n + \tfrac{1}{2} \rightsquigarrow E/\hbar\omega = 2n + 1 + \lambda$$

$$E = \hbar\omega(2n + \lambda + 1)$$

$\lambda = \sqrt{l(l+1)} = \sqrt{(l+\frac{1}{2})^2 - \frac{1}{4}}$ gives wrong result! good for large l

$= (l + \frac{1}{2})\left(1 - \frac{1}{(2l+1)^2}\right)^{1/2}$

Langer modification: $l(l+1) \longrightarrow (l + \frac{1}{2})^2$ d.h. $\lambda = l + \frac{1}{2}$

$\Rightarrow E = \hbar\omega(2n + l + 3/2)$ exact!

SUSY potential

$$\phi(r) = \sqrt{\frac{m}{2}}\,\omega r - \frac{\hbar}{\sqrt{2m}}\,\frac{\eta}{r}$$

$$\curvearrowright \quad V_\pm(r) = \frac{m}{2}\omega^2 r^2 + \frac{\hbar^2}{2m}\,\frac{\eta(\eta\pm1)}{r^2} - \hbar\omega\left(\eta\mp\tfrac{1}{2}\right)$$

shifted HO

- good SUSY : $\eta = \ell+1$, $\ell = 0,1,2,\ldots$

$$V_\pm(r) = \frac{m}{2}\omega^2 r^2 + \frac{\hbar^2}{2m}(\ell+1)(\ell+1\pm1) - \hbar\omega\left(\ell+1\mp\tfrac{1}{2}\right)$$

$$E_n^- = E_{n\ell} - \hbar\omega\left(\ell+\tfrac{3}{2}\right) = \hbar\omega\left(2n+\ell+\tfrac{3}{2}-\ell-\tfrac{3}{2}\right) = \hbar\omega\, 2n$$

$$E_n^+ = E_{n\,\ell+1} - \hbar\omega\left(\ell+\tfrac{1}{2}\right) = \hbar\omega\left(2n+\ell+1+\tfrac{3}{2}-\ell-\tfrac{1}{2}\right) = \hbar\omega\, 2(n+1)$$

- broken SUSY : $\eta = -\ell$, $\ell = 1,2,3,\ldots$

$$V_\pm(r) = \frac{m}{2}\omega^2 r^2 + \frac{\hbar^2}{2mr^2}(-\ell)(-\ell\pm1) - \hbar\omega\left(-\ell\mp\tfrac{1}{2}\right)$$

$$= \frac{m}{2}\omega^2 r^2 + \frac{\hbar^2}{2mr^2}\,\ell(\ell\mp1) + \hbar\omega\left(\ell\pm\tfrac{1}{2}\right)$$

$$E_n^- = E_{n\ell} + \hbar\omega\left(\ell-\tfrac{1}{2}\right) = \hbar\omega(2n+2\ell+1)$$

$$E_n^+ = E_{n\,\ell-1} + \hbar\omega\left(\ell+\tfrac{1}{2}\right) = \hbar\omega(2n+2\ell+1)$$

Homework:

Evaluate supersym WKB formula for above SUSY potential, i.e.

$$\int_{r_L}^{r_R} dr\,\sqrt{2m\left(E-\Phi^2(r)\right)} = \hbar\pi\left(n+\tfrac{1}{2}\pm\tfrac{\Delta}{2}\right)$$

and show that this leads to the exact eigenvalues for broken and good SUSY.

$$I := \int_{r_L}^{r_R} dr \left[2m\left(E - \frac{m}{2}\omega^2 r^2 - \frac{\hbar^2\eta^2}{2mr^2} + \hbar\omega\eta\right)\right]^{1/2}$$

same as in WKB case with $\lambda \to \eta$ and $E \to E + \hbar\omega\eta$

$$\rightsquigarrow \quad I = \frac{\pi\hbar}{2}\left[\frac{E+\hbar\omega\eta}{\hbar\omega} - |\eta|\right] \overset{!}{=} \hbar\pi\left(n + \frac{1}{2} \pm \frac{\Delta}{2}\right)$$

?

$$\rightsquigarrow \quad \frac{1}{2}\left[\frac{E}{\hbar\omega} + \eta - |\eta|\right] = n + \frac{1}{2} \pm \frac{\Delta}{2}$$

$$\rightsquigarrow \quad \underline{\underline{E = \hbar\omega\left(2n + 1 \pm \Delta - \eta + |\eta|\right)}}$$

good SUSY: $\eta = l+1 > 0$, $\Delta = 1$ $\Rightarrow$ $E_n^+ = \hbar\omega(2n+2)$, $E_n^- = \hbar\omega\, 2n$ exact!

broken SUSY: $\eta = -l < 0$, $\Delta = 0$ $\Rightarrow$ $E_n^\pm = \hbar\omega\left(2n+1+2|\eta|\right) = \hbar\omega(2n+2l+1)$

exact?