Exercise 1: N=1 SUSY of Pauli-Hamiltonian

Charged particle mass $m>0$, charge e ($e<0$ electrons)
In ext. electromagn. field

$$\vec{E} := -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\phi \qquad \text{scalar pot. } \phi$$
$$\vec{B} := \vec{\nabla}\times\vec{A} \qquad \text{vector pot. } \vec{A}$$

Minimal coupling principle:

$$i\hbar\partial_t \to i\hbar\partial_t - e\phi \;, \quad -i\hbar\vec{\nabla} \to -i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}$$

Results in SE

$$i\hbar\partial_t \psi = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 + e\phi$$

For static fields:

$$H = \frac{\vec{P}^2}{2m} \longrightarrow H_L := \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 + e\phi$$

Landau-Hamiltonian

Consider $\phi = 0$ and const. $\vec{B}$: $\vec{A} := \frac{1}{2}\vec{B}\times\vec{r}$

$$\rightsquigarrow \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 = \vec{P}^2 - \frac{e}{c}\left(\vec{P}\cdot\vec{A} + \vec{A}\cdot\vec{P}\right) + \frac{e^2}{c^2}\vec{A}^2$$

$$= \vec{P}^2 - \frac{2e}{c}\vec{A}\cdot\vec{P} + \frac{e^2}{c^2}\vec{A}^2 + \frac{ie\hbar}{c}\underbrace{(\vec{\nabla}\cdot\vec{A})}_{=0}$$

$$\vec{A}\cdot\vec{P} = \frac{1}{2}(\vec{B}\times\vec{r})\cdot\vec{P} = \frac{1}{2}(\vec{r}\times\vec{P})\cdot\vec{B} = \frac{1}{2}\vec{L}\cdot\vec{B}$$

Hence

$$H_L = \frac{\vec{P}^2}{2m} - \frac{e}{2mc}\vec{L}\cdot\vec{B} + \frac{e^2}{2mc^2}\vec{A}^2 = \frac{\vec{P}^2}{2m} - \vec{\mu}_L\cdot\vec{B} + \frac{e^2}{2mc^2}\vec{A}^2$$

magn. moment: $\vec{\mu}_L := \gamma\vec{L}$ charged particle moving on clas classical orbit with $\vec{L}$

gyromagnetic ratio: $\gamma := \frac{e}{2mc}$

We ignore the $\vec{A}^2 \sim \vec{B}^2$ term for small B and large c

$$\rightsquigarrow \quad H_L \simeq \frac{\vec{P}^2}{2m} - \vec{\mu}_L\cdot\vec{B}$$

Experiment: Energy levels for fixed ℓ doubly degenerate and $\gamma_{exp} = g\frac{e}{2mc}$ with $g \simeq 2$!

Pauli (Goudsmit and Uhlenbeck):

Postulate intrinsic spin-degree of freedom

$$\mathcal{H} = L^2(\mathbb{R}^3) \quad \longrightarrow \quad \mathcal{H} = L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$$

with $\vec{S} := \frac{\hbar}{2}\vec{\sigma}$ resulting in $\vec{\mu}_S := g\frac{e}{2mc}\vec{S}$ with $g \approx 2$

Pauli-Hamiltonian:

$$H_P = H_L - \vec{\mu}_S\cdot\vec{B} = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 + e\phi - \frac{g}{2}\frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}$$

violates minimal coupling principle !?

However on $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ we can write

$$H = \frac{\vec{P}^2}{2m} \otimes \underline{1} = \frac{(\vec{P}\cdot\vec{\sigma})^2}{2m}$$

Note $\sigma_j \sigma_k = \delta_{jk} + i \sigma_l \varepsilon_{jkl} \Rightarrow \sigma_i P_i \sigma_j P_j = P_i P_j \delta_{ij} = \vec{P}^2$

Hence minimal coupling results in

$$H_P = \frac{1}{2m}\left[(\vec{P} - \tfrac{e}{c}\vec{A})\cdot\vec{\sigma}\right]^2 + e\phi \qquad \text{with } g = 2$$

SUSY structure of H_P: Now $\phi = 0$

Supercharge: $Q_1 := (\vec{P} - \frac{e}{c}\vec{A})\cdot\vec{\sigma}\,\frac{1}{\sqrt{4m}} = Q_1^+$

$\leadsto H_P = 2Q_1^2 = \{Q_1, Q_1\}$

Explicit:

$$H_P = \frac{1}{2m}(P_i - \tfrac{e}{c}A_i)\sigma_i\,(P_k - \tfrac{e}{c}A_k)\sigma_k$$

$$= \frac{1}{2m}(\vec{P} - \tfrac{e}{c}\vec{A})^2 + \frac{i}{2m}\varepsilon_{ikl}\sigma_l\Big(\underbrace{P_i P_k}_{\to 0} - \tfrac{e}{c}(A_i P_k + A_k P_i) + \tfrac{e^2}{c^2}\underbrace{A_i A_k}_{\to 0}\Big)$$

use $P_i A_k = \frac{\hbar}{i}(\partial_i A_k) + A_k P_i$

$$= \frac{1}{2m}(\vec{P} - \tfrac{e}{c}\vec{A})^2 - \frac{e\hbar}{2mc}\underbrace{\varepsilon_{ikl}(\partial_i A_k)}_{=(\vec{\nabla}\times\vec{A})_l = B_l}\sigma_l$$

$$= \frac{1}{2m}(\vec{P} - \tfrac{e}{c}\vec{A})^2 - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} \qquad \Rightarrow g = 2!$$

SUSY requires $g = 2$

Helicity operator: $Q_1 \equiv Q$

$$\Lambda := \frac{Q}{|Q|} = \operatorname{sgn}|Q| \qquad \text{well defined on } \mathcal{H} \setminus \ker Q$$

obviously $\Lambda^2 = \mathbb{1}$ and $[H, Q] = 0 = [H, \Lambda]$

$\rightsquigarrow$ Common eigenbasis: $H|\psi_E^\pm\rangle = E|\psi_E^\pm\rangle$

$$\Lambda|\psi_E^\pm\rangle = \pm|\psi_E^\pm\rangle$$

However: Λ is <u>NOT</u> a Witten operator as we require $\{W, H\} = 0$ but $[\Lambda, H] = 0$

Velocity operator: $\vec{V} := \frac{1}{m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)$

$$\rightsquigarrow \Lambda = \frac{Q}{|Q|} = \frac{\vec{V}\cdot\vec{\sigma}}{|\vec{V}\cdot\vec{\sigma}|} \quad \text{Projection of spin on } \vec{V}$$

Remarks:

- Such a $\Lambda := \frac{Q}{|Q|}$ exists for all $N=1$ SUSY systems on $\mathcal{H} \setminus \ker Q$

  But does <u>NOT</u> generate SUSY transformations

- In $D=2$ the Pauli-Hamiltonian does allow for an $N=2$ SUSY $\Longrightarrow$ close relation to Dirac-Hamiltonian

  See later

Exercise 2: Proofs related to Witten operator

Starting point N=2 SUSY: $\{Q, Q^\dagger\} = H \;,\; Q^2 = 0 = (Q^\dagger)^2$

Auxiliary formulas: $Q Q^\dagger Q = Q(H - QQ^\dagger) = QH \overset{①}{=} HQ$

$Q^\dagger Q Q^\dagger = Q^\dagger(H - Q^\dagger Q) = Q^\dagger H \overset{①}{=} HQ^\dagger$

① Constant of motion:

$[H, Q] = [QQ^\dagger + Q^\dagger Q, Q] = QQ^\dagger Q + Q^\dagger \underbrace{QQ}_{=0} - \underbrace{QQ}_{=0} Q^\dagger - QQ^\dagger Q = 0$

$[H, Q^\dagger] = 0$ similar

② Witten parity: $[W, H] = 0 \,,\; \{W, Q\} = 0 = \{W, Q^\dagger\} \;,\; W^2 = 1$

Formal definition $W := \frac{2}{H} QQ^\dagger - 1$

a) $W = W^\dagger$ obvious as $[H, Q] = 0 = [H, Q^\dagger]$

b) $W^2 = \left(\frac{2}{H} QQ^\dagger - 1\right)^2 = \frac{4}{H^2} \underbrace{QQ^\dagger Q}_{HQ} Q^\dagger - \frac{4}{H} QQ^\dagger + 1 = 1$

c) $\{W, Q\} = \left\{\frac{2}{H} QQ^\dagger - 1, Q\right\} = \frac{2}{H} \underbrace{QQ^\dagger Q}_{HQ} - Q + \frac{2}{H} \underbrace{QQ}_{=0} Q^\dagger - Q = 0$

$\{W, Q^\dagger\} = \left\{\frac{2}{H} QQ^\dagger - 1, Q^\dagger\right\} = \frac{2}{H} Q \underbrace{Q^\dagger Q^\dagger}_{=0} - Q^\dagger + \frac{2}{H} \underbrace{Q^\dagger Q Q^\dagger}_{HQ^\dagger} - Q^\dagger = 0$

d) $[W, H] = \left[\frac{2}{H} QQ^\dagger - 1, H\right] = \frac{2}{H} QQ^\dagger H - 2H\frac{1}{H} QQ^\dagger = 0$

e) $W = \frac{2}{H} QQ^\dagger - 1 = \frac{2}{H}(H - Q^\dagger Q) - 1 = 1 - \frac{2}{H} Q^\dagger Q$

$\rightsquigarrow W = \frac{1}{H}(QQ^\dagger - Q^\dagger Q) = \frac{[Q, Q^\dagger]}{\{Q, Q^\dagger\}}$

with $Q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2) \quad Q^\dagger = \frac{1}{\sqrt{2}}(Q_1 - iQ_2)$

$QQ^\dagger = \frac{1}{2}(Q_1 + iQ_2)(Q_1 - iQ_2) = \frac{1}{2}(Q_1^2 + Q_2^2) + \frac{i}{2}(Q_2 Q_1 - Q_1 Q_2)$

$Q^\dagger Q = \frac{1}{2}(Q_1 - iQ_2)(Q_1 + iQ_2) = \frac{1}{2}(Q_1^2 + Q_2^2) - \frac{i}{2}(Q_2 Q_1 - Q_1 Q_2)$

$\Rightarrow [Q, Q^\dagger] = i[Q_2, Q_1]$

$\rightsquigarrow \quad W = \frac{1}{iH}[Q_1, Q_2]$

Exercise 3: Generalized "Fermionic" degree of freedom

Remember: $b := \frac{1}{\sqrt{H}} Q^+$, $b^+ = \frac{1}{\sqrt{H}} Q$

a) $b^2 = \frac{1}{H} Q^+ Q^+ = 0 = (b^+)^2$

b) $\{b, b^+\} = \{\frac{1}{\sqrt{H}} Q^+, \frac{1}{\sqrt{H}} Q\} = \frac{1}{H}\{Q^+, Q\} = 1$

c) $F := b^+ b = \frac{1}{H} Q Q^+ = F^+$

$F^2 = \frac{1}{H^2} \underbrace{Q Q^+ Q}_{HQ} Q^+ = \frac{1}{H} Q Q^+ = F$

$\rightsquigarrow$ spec $F = \{0, 1\}$

d) $W = \frac{2}{H} Q Q^+ - 1 = 2F - 1 = (-1)^{F+1}$

$\rightsquigarrow$ spec $W = \{-1, 1\}$

Boson Fermion

Matrix representation:

$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ $Q^+ = \begin{pmatrix} 0 & 0 \\ A^+ & 0 \end{pmatrix}$ $H = \begin{pmatrix} AA^+ & 0 \\ 0 & A^+A \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$

$\rightsquigarrow \frac{1}{H} = \begin{pmatrix} \frac{1}{H_+} & 0 \\ 0 & \frac{1}{H_-} \end{pmatrix}$ $\rightsquigarrow W = \frac{2}{H} Q Q^+ - 1 = \frac{2}{H}\begin{pmatrix} AA^+ & 0 \\ 0 & 0 \end{pmatrix} - 1 = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$F = \frac{1}{H} Q Q^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Exercise 4: On the matrix reps. of $N=2$ SUSY systems

- Witten operator in eigen basis:

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{array}{l} \leftarrow \mathcal{H}^+ \\ \leftarrow \mathcal{H}^- \end{array} \qquad \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

- General ansatz for supercharge:

$$Q = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \quad \text{where} \quad \begin{array}{ll} \alpha, \alpha^\dagger : \mathcal{H}^+ \to \mathcal{H}^+ & A, B^\dagger : \mathcal{H}^- \to \mathcal{H}^+ \\ \beta, \beta^\dagger : \mathcal{H}^- \to \mathcal{H}^- & A^\dagger, B : \mathcal{H}^+ \to \mathcal{H}^- \end{array}$$

a) $$WQ = \begin{pmatrix} \alpha & A \\ -B & -\beta \end{pmatrix} \qquad QW = \begin{pmatrix} \alpha & -A \\ B & -\beta \end{pmatrix}$$

with $\{W, Q\} \overset{!}{=} 0 \Rightarrow \underline{\underline{\alpha = 0}}$ and $\underline{\underline{\beta = 0}}$

b) $$0 \overset{!}{=} Q^2 = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}$$

$$0 \overset{!}{=} (Q^\dagger)^2 = \begin{pmatrix} 0 & B^\dagger \\ A^\dagger & 0 \end{pmatrix}\begin{pmatrix} 0 & B^\dagger \\ A^\dagger & 0 \end{pmatrix} = \begin{pmatrix} B^\dagger A^\dagger & 0 \\ 0 & A^\dagger B^\dagger \end{pmatrix}$$

$$\boxed{\begin{array}{c} AB = 0 = BA \\ A^\dagger B^\dagger = 0 = B^\dagger A^\dagger \end{array}}$$

c) $$H = \{Q, Q^\dagger\} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 & B^\dagger \\ A^\dagger & 0 \end{pmatrix} + \begin{pmatrix} 0 & B^\dagger \\ A^\dagger & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} AA^\dagger + B^\dagger B & 0 \\ 0 & BB^\dagger + A^\dagger A \end{pmatrix}$$

$\rightsquigarrow$ System decouples into 2 independent $N=2$ SUSY systems "A" and "B"

Let $$H_A := \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} \qquad H_B := \begin{pmatrix} B^\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix} \qquad H = H_A + H_B$$

$$Q_A := \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \qquad Q_B := \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \qquad Q = Q_A + Q_B$$

Homework: Proof that all A-operators commute with all B-operators

$\Rightarrow \{H, Q, Q^\dagger, W; \mathcal{H}\}$ decomposes into subsystems (independent)

$\{H_A, Q_A, Q_A^\dagger, W_A; \mathcal{H}\}$ and $\{H_B, Q_B, Q_B^\dagger, W_B; \mathcal{H}\}$

$$W_A := \frac{2}{H_A} Q_A Q_A^\dagger - 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = W$$

$$2 Q_B Q_B^\dagger - H_B = 2\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 & B^\dagger \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} B^\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix} = 2\begin{pmatrix} 0 & 0 \\ 0 & BB^\dagger \end{pmatrix} - \begin{pmatrix} B^\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix} = \begin{pmatrix} -B^\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix}$$

$$= H_B \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -H_B W \Rightarrow W_B := \frac{2}{H_B} Q_B Q_B^\dagger - 1 = -W$$

System A	$\longleftrightarrow$	System B
A		$B^\dagger$
H_A		H_B
Q_A		$Q_B^\dagger$
$Q_A^\dagger$		Q_B
W		$-W$