8 Supersymmetry in the Dirac-Hamiltonian

8.1 The Dirac equation

Problem: (see e.g. F. Schwabl, "QM für Fortgeschrittene")
Schrödinger eq. allows for a probabilistic interpretation but is no *relativistic* description. Klein-Gordon eq. $E^2 = \vec{p}^2c^2 + m^2c^4$ is covariant and relativistic, but does not allow for a probabilistic interpretation.

Dirac's ansatz:

$$H := c\vec{\alpha}\cdot\vec{p} + \beta mc^2$$

with $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T$ and β being arbitrary, not necessarily, numbers.

Quantisation: $E \to H$ and $\vec{p} \to -\mathrm{i}\hbar\vec{\nabla}$ results in

$$\begin{aligned} H^2 &= -c^2\hbar^2\alpha_k\alpha_l\partial_k\partial_l - \mathrm{i}\hbar mc^2(\alpha_k\beta + \beta\alpha_k)\partial_k + \beta^2m^2c^4 \\ &= -\tfrac{1}{2}c^2\hbar^2(\alpha_k\alpha_l + \alpha_l\alpha_k)\partial_k\partial_l - \mathrm{i}\hbar mc^2(\alpha_k\beta + \beta\alpha_k)\partial_k + \beta^2m^2c^4 \end{aligned}$$

Compare with KG relation $E^2 = \vec{p}^2c^2 + m^2c^4$ led Dirac to the conclusion

$$\left.\begin{aligned} \{\alpha_k, \alpha_l\} &= 2\delta_{kl} \\ \{\alpha_k, \beta\} &= 0 \\ \beta^2 &= 1 \end{aligned}\right\} \quad \text{Dirac matrices, Dirac algebra}$$

Further properties: $H = H^\dagger \quad \Longrightarrow \quad \alpha_k = \alpha_k^\dagger \quad$ and $\quad \beta = \beta^\dagger$

Consider: $\mathrm{Tr}\,\alpha_k = \mathrm{Tr}\,\alpha_k\beta^2 = -\mathrm{Tr}\,\beta\alpha_k\beta = -\mathrm{Tr}\,\alpha_k \quad \Longrightarrow \quad \mathrm{Tr}\,\alpha_k = 0$

Similar $\mathrm{Tr}\,\beta = \mathrm{Tr}\,\beta\alpha_k^2 = -\mathrm{Tr}\,\alpha_k\beta\alpha_k = -\mathrm{Tr}\,\beta \quad \Longrightarrow \quad \mathrm{Tr}\,\beta = 0$

- Pauli representation: 4×4-matrices

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \text{or} \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Weyl representation:

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{via} \quad U_W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- Supersymmetric representation:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{via} \quad U_S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$$

- Free Dirac equation: $H = \mathrm{i}\hbar\partial_t$

$$\boxed{\mathrm{i}\hbar\partial_t\Psi(\vec{r},t) = (c\vec{\alpha}\cdot\vec{p} + \beta mc^2)\Psi(\vec{r},t)}$$

Ψ: Dirac spinor, lives in $\mathcal{H} = L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$

$$\Psi(\vec{r},t) = \begin{pmatrix} \psi_1(\vec{r},t) \\ \psi_2(\vec{r},t) \\ \psi_3(\vec{r},t) \\ \psi_4(\vec{r},t) \end{pmatrix}$$

- Free Dirac Hamiltonian: (Pauli representation)

$$\boxed{H_0 := c\vec{\alpha}\cdot\vec{p} + \beta mc^2 = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{pmatrix}}$$

- Free massless Dirac Hamiltonian: (Weyl representation)

$$H_0 = c\vec{\alpha}\cdot\vec{p} = \begin{pmatrix} c\vec{\sigma}\cdot\vec{p} & 0 \\ 0 & -c\vec{\sigma}\cdot\vec{p} \end{pmatrix}$$

$\Longrightarrow$ Weyl eq. $\mathrm{i}\hbar\partial_t\Psi = c\vec{\sigma}\cdot\vec{p}\Psi\,, \qquad \Psi \in L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$

- Free Dirac Hamiltonian in 1D and 2D:

$$H_0 = -\mathrm{i}\hbar c\,(\sigma_1\partial_1 + \sigma_2\partial_2) + \sigma_3 mc^2$$

- Charged Dirac particle in electromagnetic potentials:
via minimal coupling $\vec{p} \to \vec{p} - \frac{e}{c}\vec{A}$ and $\mathrm{i}\hbar\partial_t \to \mathrm{i}\hbar\partial_t - e\phi_{el}$

$$\boxed{H = c\vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) + \beta mc^2 + e\phi_{el}}$$

- Scalar potentials: $V(\vec{r}) = \beta\phi_{sc}(\vec{r})$

- Dirac oscillator:

$$H = c\vec{\alpha}\cdot(\vec{p} + \beta\mathrm{i}m\omega\vec{r}) + \beta mc^2 = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot(\vec{p} - \mathrm{i}m\omega\vec{r}) \\ c\vec{\sigma}\cdot(\vec{p} + \mathrm{i}m\omega\vec{r}) & -mc^2 \end{pmatrix}$$

More details: B. Thaller, "The Dirac Equation" (Springer, Berlin, 1992)

8.2 Supersymmetric Dirac operators

Recall: $N = 2$ SUSY with Witten operator now on $\mathcal{H} = L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

SUSY Hamiltonian:

$$H_S := \{Q, Q^\dagger\} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

Definition:
Let

$$Q_1 := Q + Q^\dagger = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} = Q_1^\dagger \qquad \text{and} \qquad \mathcal{M} := \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} = \mathcal{M}^\dagger \geq 0$$

then

$$\boxed{H_D := Q_1 + \mathcal{M}W}$$

is called *supersymmetric Dirac operator* if $[Q_1, \mathcal{M}] = 0 = [W, \mathcal{M}]$.
That is,

$$\boxed{H_D = \begin{pmatrix} M_+ & A \\ A^\dagger & -M_- \end{pmatrix} \qquad \text{with} \qquad AM_- = M_+A\,, \qquad A^\dagger M_+ = M_-A^\dagger\,.}$$

Example: $A := c\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A}) = A^\dagger\,, \qquad M_\pm = mc^2 \otimes 1$

$$H_D = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A}) \\ c\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A}) & -mc^2 \end{pmatrix} = c\vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) + \beta mc^2$$

Charged Dirac particle in magnetic field.

Properties:

- Consider

$$\begin{aligned} H_D^2 &= (Q_1 + \mathcal{M}W)^2 = \begin{pmatrix} M_+ & A \\ A^\dagger & -M_- \end{pmatrix} \begin{pmatrix} M_+ & A \\ A^\dagger & -M_- \end{pmatrix} \\ &= \begin{pmatrix} AA^\dagger + M_+^2 & M_+A - AM_- \\ A^\dagger M_+ - M_-A^\dagger & A^\dagger A + M_-^2 \end{pmatrix} = \begin{pmatrix} AA^\dagger + M_+^2 & 0 \\ 0 & A^\dagger A + M_-^2 \end{pmatrix} \end{aligned}$$

  Let $m > 0$ be an arbitrary mass-like parameter and define

$$H_+ := \frac{1}{2mc^2} AA^\dagger \,, \qquad H_- := \frac{1}{2mc^2} A^\dagger A \,,$$

  Rescale supercharges

$$Q := \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} , \qquad Q^\dagger = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}$$

  and set

$$H_{SUSY} := \frac{1}{2mc^2} \left(H_D^2 - \mathcal{M}^2 \right) = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} .$$

  Then we obtain a $N = 2$ SUSY QM system with $W = \beta$

$$H_{SUSY} = \{Q, Q^\dagger\} \,, \qquad \{Q, W\} = 0 \,, \qquad Q^2 = 0 = (Q^\dagger)^2 \,.$$

- Let $U_{FW} := a_+ + W \operatorname{sgn} Q_1 \, a_-$ be unitary transformation with $a_\pm := \sqrt{\frac{1}{2} \pm \frac{\mathcal{M}}{2|H_D|}}$
  Then (see tutorial)

$$H_D^{FW} := U_{FW} H_D U_{FW}^\dagger = \begin{pmatrix} \sqrt{AA^\dagger + M_+^2} & 0 \\ 0 & -\sqrt{A^\dagger A + M_-^2} \end{pmatrix} = \beta |H_D|$$

  U_{FW} diagonalises H_D and is called *Foldy–Wouthuysen transformation.*

  Hence with

$$H_D^{FW} \widetilde{\Psi_n^\pm} = E_n^\pm \widetilde{\Psi_n^\pm} \qquad \text{and} \qquad \Psi_n^\pm := U_{FW}^\dagger \widetilde{\Psi_n^\pm} \qquad \Longrightarrow \qquad H_D \Psi_n^\pm = E_n^\pm \Psi_n^\pm$$

  The subspaces $\mathcal{H}^\pm$ are the eigenspaces of H_D for positive and negative energies, respectively.

  Observation: In many cases $M_\pm = mc^2$ and $A = A^\dagger$, that is $H_{NR} := H_\pm = \frac{A^2}{2mc^2}$

$$H_D^{FW} = \beta mc^2 \sqrt{1 + \frac{H_{NR}}{2mc^2}}$$

  Hence H_{NR} is the non-relativistic limit of H_D in those cases as

$$H_D^{FW}\big|_{\mathcal{H}^+} = mc^2 + H_{NR} + O\left(1/mc^2\right)$$

- Spectral properties: Note $[H_+, M_+] = 0 = [H_-, M_-]$

  Let $\quad H_\pm \phi_n^\pm = \varepsilon_n \phi_n^\pm \quad$ and $\quad M_\pm \phi_n^\pm = m_n c^2 \phi_n^\pm \quad$ with $\quad \varepsilon_n, m_n > 0$

  Hence we have

$$\boxed{E_n^\pm = \pm\sqrt{2mc^2 \varepsilon_n + m_n c^2} \,, \qquad \widetilde{\Psi_n^+} = \begin{pmatrix} \phi_n^+ \\ 0 \end{pmatrix} , \qquad \widetilde{\Psi_n^-} = \begin{pmatrix} 0 \\ \phi_n^- \end{pmatrix}}$$

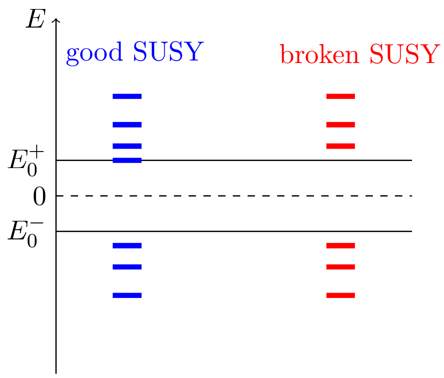
For unbroken SUSY ($\varepsilon_0 = 0$) in addition we have

$$E_0^+ = \langle \phi_0^+ | M_+ | \phi_0^+ \rangle \qquad \text{if} \qquad \phi_0^+ \in \mathcal{H}^+ \text{exists with} \qquad A^\dagger \phi_0^+ = 0$$

and/or

$$E_0^- = -\langle \phi_0^- | M_- | \phi_0^- \rangle \qquad \text{if} \qquad \phi_0^- \in \mathcal{H}^- \text{exists with} \qquad A \phi_0^- = 0$$

The spectrum of a supersymmetric Dirac Hamiltonian is symmetric about zero with the exception at E_0^+ and/or E_0^- if SUSY is unbroken.



The spectral properties of H_D follow from those of the SUSY partners $H_\pm$ and $M_\pm$. In all most all case, $M_\pm = mc^2$ or $M_\pm = 0$.

Note: In general $A \sim \vec{p}$, hence $H_\pm \sim \vec{p}^2$, i.e. the relativistic problem may be reduced to a non-relativistic Pauli-like problem.

Example: Electron in magnetic field results in $H_D^{FW} = \beta mc^2 \sqrt{1 + \frac{2H_P}{mc^2}}$

Dirac: $\qquad H_D = c\vec{\alpha} \cdot \left(\vec{p} - \frac{e}{c}\vec{A}\right) + \beta mc^2$

Pauli: $\qquad H_P = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{2mc}\vec{\sigma} \cdot \vec{B}$

- SUSY transformations for $\varepsilon_n > 0$:

  Recall $\qquad \phi_n^+ = \dfrac{1}{\sqrt{2mc^2\varepsilon_n}} A\phi_n^- \qquad$ and $\qquad \phi_n^- = \dfrac{1}{\sqrt{2mc^2\varepsilon_n}} A^\dagger \phi_n^+$.

  Hence $\qquad \widetilde{\Psi_n^+} = \dfrac{1}{\sqrt{\varepsilon_n}} Q\widetilde{\Psi_n^-} \qquad$ and $\qquad \widetilde{\Psi_n^-} = \dfrac{1}{\sqrt{\varepsilon_n}} Q^\dagger \widetilde{\Psi_n^+}$

  Obvious as

$$Q\,\widetilde{\Psi_n^-} = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \phi_n^- \end{pmatrix} = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} A\phi_n^- \\ 0 \end{pmatrix} = \sqrt{\varepsilon_n}\,\widetilde{\Psi_n^+}$$

$$Q^\dagger\,\widetilde{\Psi_n^+} = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix} \begin{pmatrix} \phi_n^+ \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 \\ A^\dagger \phi_n^+ \end{pmatrix} = \sqrt{\varepsilon_n}\,\widetilde{\Psi_n^-}$$

8.3 The free Dirac Hamiltonian

Choose: $\qquad A := c\vec{\sigma} \cdot \vec{p} = A^\dagger\,, \qquad M_\pm := mc^2 \qquad$ on $\qquad \mathcal{H}^\pm = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$

$$H_D = \begin{pmatrix} mc^2 & c\vec{\sigma} \cdot \vec{p} \\ c\vec{\sigma} \cdot \vec{p} & -mc^2 \end{pmatrix} \qquad \text{Pauli reps.}$$

With $A^\dagger A = c^2(\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) = c^2\vec{p}^2 = AA^\dagger$ we have

$$H_\pm = \frac{1}{2mc^2} c^2 \vec{p}^2 = \frac{\vec{p}^2}{2m} \qquad \text{free non-rel. particle on } \mathcal{H}^\pm$$

$\varepsilon_0 = 0 \in \operatorname{spec} H_\pm \qquad \Longrightarrow \qquad$ SUSY unbroken

Eigenspinors: Plane waves

$$\phi^\pm_{\vec{k}\lambda}(\vec{r}) = \left(\frac{1}{2\pi\hbar}\right)^{3/2} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}}\ \chi_\lambda(\vec{k})\,, \qquad \vec{k} \in \mathbb{R}^3\,, \qquad \lambda \in \{-1,+1\}\,,$$

with eigenvalues $\varepsilon_{\vec{k}} = \dfrac{\hbar^2\vec{k}^{\,2}}{2m}$ and 2-spinor $\chi_\lambda(\vec{k})$

Helicity eigenspinors: Let $k := |\vec{k}|$ and

$$\chi_{+1}(\vec{k}) := \frac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k_1 - \mathrm{i}k_2 \\ k - k_3 \end{pmatrix} \qquad \text{and} \qquad \chi_{+1}(k\vec{e}_3) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,,$$

$$\chi_{-1}(\vec{k}) := \frac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k_3 - k \\ k_1 + \mathrm{i}k_2 \end{pmatrix} \qquad \text{and} \qquad \chi_{-1}(k\vec{e}_3) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,,$$

Recall helicity operator $\Lambda := \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$ in eigenspace with fixed $\vec{k}$, $\Lambda_{\vec{k}} := \dfrac{\vec{\sigma}\cdot\vec{k}}{k}$

Lemma: Above spinors are ortho-normal eigenspinors of $\Lambda_{\vec{k}}$, that is,

$$\Lambda_{\vec{k}}\chi_\lambda(\vec{k}) = \lambda\chi_\lambda(\vec{k})\,, \qquad \lambda = \pm 1\,, \qquad ||\chi_\lambda||^2 = (\chi^*_\lambda)^T\chi_\lambda = 1\,, \qquad (\chi^*_{-1})^T\chi_{+1} = 0\,.$$

Proof:
Consider $\vec{\sigma}\cdot\vec{k} = \begin{pmatrix} k_3 & k_1 - \mathrm{i}k_2 \\ k_1 + \mathrm{i}k_2 & -k_3 \end{pmatrix} \qquad \Longrightarrow$

$$\begin{aligned}
\vec{\sigma}\cdot\vec{k}\chi_{+1}(\vec{k}) &= \tfrac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k_3 & k_1 - \mathrm{i}k_2 \\ k_1 + \mathrm{i}k_2 & -k_3 \end{pmatrix}\begin{pmatrix} k_1 - \mathrm{i}k_2 \\ k - k_3 \end{pmatrix} \\
&= \tfrac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k_3(k_1 - \mathrm{i}k_2) + (k_1 - \mathrm{i}k_2)(k - k_3) \\ k_1^2 + k_2^2 - k_3(k - k_3) \end{pmatrix} \\
&= \tfrac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k(k_1 - \mathrm{i}k_2) \\ k(k - k_3) \end{pmatrix} = k\chi_{+1}(\vec{k})
\end{aligned}$$

$$\begin{aligned}
\vec{\sigma}\cdot\vec{k}\chi_{-1}(\vec{k}) &= \tfrac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k_3 & k_1 - \mathrm{i}k_2 \\ k_1 + \mathrm{i}k_2 & -k_3 \end{pmatrix}\begin{pmatrix} k_3 - k \\ k_1 + \mathrm{i}k_2 \end{pmatrix} \\
&= \tfrac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k_3(k_3 - k) + k_1^2 + k_2^2 \\ (k_1 + \mathrm{i}k_2)(k_3 - k) - k_3(k_1 + \mathrm{i}k_2) \end{pmatrix} \\
&= \tfrac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k(k - k_3) \\ -k(k_1 + \mathrm{i}k_2) \end{pmatrix} = -k\chi_{-1}(\vec{k})
\end{aligned}$$

The ortho-normal part is homework.

Summary:

$$H_\pm\ \phi^\pm_{\vec{k}\lambda}(\vec{r}) = \varepsilon_{\vec{k}}\ \phi^\pm_{\vec{k}\lambda}(\vec{r}) \qquad \text{with} \qquad \varepsilon_{\vec{k}} = \tfrac{\hbar^2\vec{k}^{\,2}}{2m}\,, \quad \vec{k} \in \mathbb{R}^3\,,$$

$$\Lambda\ \phi^\pm_{\vec{k}\lambda}(\vec{r}) = \lambda\ \phi^\pm_{\vec{k}\lambda}(\vec{r}) \qquad \text{with} \qquad \lambda = \pm 1\,.$$

$$\boxed{E^\pm_{\vec{k}\lambda} = \pm\sqrt{\hbar^2c^2\vec{k}^{\,2} + m^2c^4}\,, \qquad \widetilde{\Psi^+_{\vec{k}\lambda}}(\vec{r}) = \begin{pmatrix} \phi^+_{\vec{k}\lambda}(\vec{r}) \\ 0 \end{pmatrix}\,, \qquad \widetilde{\Psi^-_{\vec{k}\lambda}}(\vec{r}) = \begin{pmatrix} 0 \\ \phi^-_{\vec{k}\lambda}(\vec{r}) \end{pmatrix}}$$

Explicit form of FW transformation:
Consider subspace with fixed $\vec{k}$ and λ and set $\epsilon(k) := \sqrt{\hbar^2c^2k^2 + m^2c^4}$

- With $\quad |H_D|\widetilde{\Psi^{\pm}_{\vec{k}\lambda}} = \epsilon(k)\widetilde{\Psi^{\pm}_{\vec{k}\lambda}} \quad \Longrightarrow \quad a_\pm = \sqrt{\frac{1}{2} \pm \frac{mc^2}{2\epsilon(k)}}$

- $\operatorname{sgn} Q_1 = \frac{Q_1}{\sqrt{Q_1^2}}, \quad Q_1 = \begin{pmatrix} 0 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}, \quad Q_1^2 = \begin{pmatrix} c^2\vec{p}^2 & 0 \\ 0 & c^2\vec{p}^2 \end{pmatrix} = c^2\vec{p}^2 \otimes \mathbf{1}$
  $\operatorname{sgn} Q_1 = \frac{1}{\sqrt{c^2\vec{p}^2}} \begin{pmatrix} 0 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Lambda \\ \Lambda & 0 \end{pmatrix} = \Lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- $W \operatorname{sgn} Q_1 = \Lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

- $U_{FW} = a_+ + W \operatorname{sgn} Q_1 a_- = a_+ + a_-\Lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U_{FW}^\dagger = a_+ - a_-\Lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

- The "electron" solution
  $\Psi^+_{\vec{k}\lambda}(\vec{r}) = U^\dagger_{FW}\widetilde{\Psi^+_{\vec{k}\lambda}}(\vec{r}) = \left[a_+ - \lambda a_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \begin{pmatrix} \phi^+_{\vec{k}\lambda}(\vec{r}) \\ 0 \end{pmatrix}$

$$\boxed{\Psi^+_{\vec{k}\lambda}(\vec{r}) = \begin{pmatrix} a_+\phi^+_{\vec{k}\lambda}(\vec{r}) \\ \lambda a_-\phi^+_{\vec{k}\lambda}(\vec{r}) \end{pmatrix}}$$

- The "positron" solution

$$\boxed{\Psi^-_{\vec{k}\lambda}(\vec{r}) = \begin{pmatrix} -\lambda a_-\phi^-_{\vec{k}\lambda}(\vec{r}) \\ a_+\phi^-_{\vec{k}\lambda}(\vec{r}) \end{pmatrix}}$$

- SUSY transformations: $A = c\vec{\sigma}\cdot\vec{p} = A^\dagger \;, \varepsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m}$
  $A\,\phi^\pm_{\vec{k}\lambda} = c\hbar\vec{\sigma}\cdot\vec{k}\;\phi^\pm_{\vec{k}\lambda} = c\hbar k\lambda\;\phi^\mp_{\vec{k}\lambda} = \lambda\sqrt{2mc^2\varepsilon_{\vec{k}}}\;\phi^\mp_{\vec{k}\lambda}$ (λ is phase only!)
  $Q^\dagger\widetilde{\Psi^+_{\vec{k}\lambda}} = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix} \begin{pmatrix} \phi^+_{\vec{k}\lambda} \\ 0 \end{pmatrix} = \lambda\sqrt{\varepsilon_{\vec{k}}} \begin{pmatrix} 0 \\ \phi^-_{\vec{k}\lambda} \end{pmatrix} = \lambda\sqrt{\varepsilon_{\vec{k}}}\,\widetilde{\Psi^-_{\vec{k}\lambda}}$
  $Q\widetilde{\Psi^-_{\vec{k}\lambda}} = \cdots = \lambda\sqrt{\varepsilon_{\vec{k}}}\,\widetilde{\Psi^+_{\vec{k}\lambda}}$

- Free Dirac particle in SUSY representation:
  Now $\quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad$ and $\quad \beta = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$

$$H_D = c\vec{\alpha}\cdot\vec{p} + \beta mc^2 = \begin{pmatrix} 0 & c\vec{\sigma}\cdot\vec{p} - \mathrm{i}mc^2 \\ c\vec{\sigma}\cdot\vec{p} + \mathrm{i}mc^2 & 0 \end{pmatrix}$$

  Hence

$$A := c\vec{\sigma}\cdot\vec{p} - \mathrm{i}mc^2 \qquad \text{and} \qquad M_\pm := 0 \qquad \Longrightarrow$$

$$H_+ = \frac{AA^\dagger}{2mc^2} = \frac{A^\dagger A}{2mc^2} = H_- \qquad \text{or} \qquad H_\pm = \frac{c^2\vec{p}^2}{2mc^2} + \frac{m^2c^4}{2mc^2} \geq \frac{1}{2}mc^2 > 0$$

  Here SUSY is broken with

$\varepsilon_{\vec{k}} = \frac{\hbar^2\vec{k}^2}{2m} + \frac{1}{2}mc^2$	shifted SUSY spectrum
$\phi^\pm_{\vec{k}\lambda}(\vec{r}) = \left(\frac{1}{2\pi\hbar}\right)^{3/2} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}}\;\chi_\lambda(\vec{k})$	same eigenspinors
$E^\pm_{\vec{k}} = \pm\sqrt{2mc^2\varepsilon_{\vec{k}}} = \pm\sqrt{c^2\hbar^2k^2 + m^2c^4}$	same Dirac spectrum

8.4 The Dirac oscillator

$$H = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot(\vec{p}-\mathrm{i}m\omega\vec{r}) \\ c\vec{\sigma}\cdot(\vec{p}+\mathrm{i}m\omega\vec{r}) & -mc^2 \end{pmatrix}$$

Is obviously SUSY Dirac Hamiltonian

$$A = c\vec{\sigma}\cdot(\vec{p}-\mathrm{i}m\omega\vec{r})\,, \qquad M_\pm = mc^2 \qquad \Longrightarrow \qquad AM_- = M_+A\,, \qquad A^\dagger M_+ = M_-A^\dagger$$

Homework: Show

$$\begin{aligned} AA^\dagger &= c^2\left(\vec{p}^{\,2} + m^2\omega^2\vec{r}^{\,2} + 3mc^3\hbar\omega + 2mc^2\omega\vec{L}\cdot\vec{\sigma}\right) = 2mc^2 H_+ \\ A^\dagger A &= c^2\left(\vec{p}^{\,2} + m^2\omega^2\vec{r}^{\,2} - 3mc^3\hbar\omega - 2mc^2\omega\vec{L}\cdot\vec{\sigma}\right) = 2mc^2 H_- \end{aligned}$$

Partner Hamitonians are SUSY Pauli Hamiltonians

$$\begin{aligned} H_\pm &= \frac{\vec{p}^{\,2}}{2m} + \frac{m}{2}\omega^2\vec{r}^{\,2} \pm \left(\frac{3}{2}\hbar\omega + \hbar\omega\vec{L}\cdot\vec{\sigma}\right) \\ &= \frac{\vec{p}^{\,2}}{2m} + \frac{m}{2}\omega^2\vec{r}^{\,2} \pm \hbar\omega\left(K + \frac{1}{2}\right) \end{aligned}$$

Recall spin orbit operator $K := \vec{L}\cdot\vec{\sigma} + 1$

Eigenvalues of K are given by: $-\kappa = s|\kappa| = s(j+\frac{1}{2}) = \begin{cases} \ell+1 & \text{for} \quad s=+1 \\ -\ell & \text{for} \quad s=-1 \end{cases}$ or $j = \ell + \frac{s}{2}$

Eigenvalues of $H_\pm$:

$$\varepsilon^\pm_{njs} = \hbar\omega\left(2n + \ell + \tfrac{3}{2}\right) \pm \hbar\omega\left[s(j+\tfrac{1}{2}) + \tfrac{1}{2}\right]$$

More explicit

$$\begin{aligned} \varepsilon^-_{njs} &= \hbar\omega\left(2n + j - \tfrac{s}{2} + \tfrac{3}{2} - sj - \tfrac{s}{2} - \tfrac{1}{2}\right) = \hbar\omega\left[2n + j + 1 - s(j+1)\right] \\ \varepsilon^+_{njs} &= \hbar\omega\left(2n + j - \tfrac{s}{2} + \tfrac{3}{2} + sj + \tfrac{s}{2} + \tfrac{1}{2}\right) = \hbar\omega\left[2(n+1) + j + sj\right] > 0 \end{aligned}$$

SUSY unbroken with ground state energy

$$\varepsilon^-_{0j1} = 0 \qquad \infty\text{−degenerate as} \qquad j = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots$$

Spectral relation between SUSY partners

$$\varepsilon^+_{njs} = \varepsilon^-_{n+1,j-1,-s}$$

Eigenvalues of the Dirac oscillator

$$E^-_{njs} = -mc^2\left[1 + \frac{2\hbar\omega}{mc^2}\left[2n + j + 1 - s(j+1)\right]\right]^{1/2}$$

$$E^+_{njs} = mc^2\left[1 + \frac{2\hbar\omega}{mc^2}\left[2(n+1) + j + sj\right]\right]^{1/2}$$

8.5 One-dimensional Dirac Hamiltonians

- The free Dirac particle on the real line

$$H = c\sigma_1 p + \sigma_3 mc^2 = \begin{pmatrix} mc^2 & cp \\ cp & -mc^2 \end{pmatrix} \qquad \text{on} \qquad \mathcal{H} = L^2(\mathbb{R})\otimes\mathbb{C}^2$$

Obvious: $\quad A = cp = A^\dagger\,, \qquad M_\pm = mc^2\,, \qquad H_\pm = \frac{A^2}{2mc^2} = \frac{p^2}{2m} \geq 0$

- The Dirac oscillator on the real line

$$H = c\sigma_1(p + \mathrm{i}m\omega x\sigma_3) + \sigma_3 mc^2 = \begin{pmatrix} mc^2 & c(p - \mathrm{i}m\omega x) \\ c(p + \mathrm{i}m\omega x) & -mc^2 \end{pmatrix}$$

Obvious: $\qquad A = c(p - \mathrm{i}m\omega x) = -\mathrm{i}\sqrt{2mc^2\hbar\omega}\, a\,, \qquad M_\pm = mc^2$

$$H_\pm = \tfrac{p^2}{2m} + \tfrac{m}{2}\omega^2 x^2 \pm \tfrac{1}{2}\hbar\omega = \hbar\omega\left(a^\dagger a + \tfrac{1}{2} \pm \tfrac{1}{2}\right)$$

Hence

$$\phi_n^+ = \langle x|n-1\rangle\,, \qquad \phi_n^- = \langle x|n\rangle\,, \qquad \varepsilon_n := \hbar\omega n\,, \qquad n = 1, 2, 3, \ldots$$

in addition $n = 0$ for H_- only.

$$E_n^\pm = \pm\sqrt{m^2c^4 + 2mc^2\varepsilon_n} = \pm mc^2\sqrt{1 + \frac{2\varepsilon_n}{mc^2}}$$

- The relativistic Witten model
  Generalisation of Dirac oscillator with $m\omega x \to \sqrt{2m}\Phi(x)$

$$H = c\sigma_1(p + \mathrm{i}\sqrt{2m}\Phi(x)\sigma_3) + \sigma_3 mc^2 = \begin{pmatrix} mc^2 & c(p - \mathrm{i}\sqrt{2m}\Phi(x)) \\ c(p + \mathrm{i}\sqrt{2m}\Phi(x)) & -mc^2 \end{pmatrix}$$

Obvious: $\qquad A = c(p - \mathrm{i}\sqrt{2m}\Phi(x))\,, \qquad M_\pm = mc^2$

$$H_\pm = \frac{p^2}{2m} + \Phi(x)^2 \pm \frac{\hbar}{\sqrt{2m}}\Phi'(x)$$

Assume unbroken SUSY with $\varepsilon_0 = 0 \in \operatorname{spec} H_-$ and $\varepsilon_n > 0 \in \operatorname{spec} H_+$ then

$$E_0^- = -mc^2 \qquad \text{and} \qquad E_n^\pm = \pm mc^2\sqrt{1 + \frac{2\varepsilon_n}{mc^2}}$$

Remarks:

  - Whenever the non-relativistic Witten model can be solved, one also has a solution of the relativistic Witten model.
  - Application of the SUSY WKB formula results in an approximation for the relativistic Witten model via $E^2 = 2mc^2\varepsilon + m^2c^4$.
    Let $W(x) := \sqrt{2mc^2}\,\Phi(x)$, then $A = cp - \mathrm{i}W(x)$ and

$$\int_{x_L}^{x_R} \mathrm{d}x\sqrt{E^2 - m^2c^4 - W^2(x)} = c\hbar\pi\left(n + \frac{1}{2} \pm \frac{\Delta}{2}\right)$$

    with $W^2(x_{R/L}) = E^2 - m^2c^4$.

For a general discussion see GJ, Eur. Phys. J. Plus 135 (2020) 464 (13pp)

8.6 Relativistic Hamiltonians with arbitray spin

The Dirac Hamiltonian describes the relativistic dynamics of spin-$\frac{1}{2}$ particles.
How about particles with other spin?
Goal is to find relativistic eq. allowing for a probability interpretation, that is, being of the form

$$\mathrm{i}\hbar\partial_t\Psi = H\Psi\,, \qquad \mathcal{H} = L^2(\mathbb{R}^3)\otimes\mathbb{C}^{2(2s+1)}\,, \qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

The general form of such a Hamiltonian is given by

$$H = \beta m + \mathcal{E} + \mathcal{O}\,, \qquad \text{with} \qquad \beta^2 = 1\,.$$

Here m denotes the mass of the particle.
$\mathcal{E}$ and $\mathcal{O}$ denote the *even* and *odd* parts of the Hamiltonian, respectively. That is,

$$[\beta, \mathcal{E}] = 0\,, \qquad \{\beta, \mathcal{O}\} = 0\,.$$

With $\mathcal{M} := m + \beta\mathcal{E}$ the general Hamiltonian then reads

$$H_s = \beta\mathcal{M} + \mathcal{O} \qquad \text{with} \qquad \begin{array}{llll} H_s = H_s^\dagger & \text{for} & s = \frac{1}{2}, \frac{3}{2}, \ldots\,, & \textit{Fermions} \\ H_s = \beta H_s^\dagger \beta & \text{for} & s = 0, 1, 2, \ldots\,, & \textit{Bosons} \end{array}$$

Choose matrix representation where

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{M} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}, \quad \mathcal{O} = \begin{pmatrix} 0 & A \\ (-1)^{2s+1}A^\dagger & 0 \end{pmatrix},$$

Note: The matrix elements here are $(2s + 2) \times (2s + 2)$ submatrices.

Definition:
Above Hamiltonian H_s is called a *supersymmetric relativistic arbitrary-spin Hamiltonian* if

$$M_+ A = A M_-\,, \qquad A^\dagger M_+ = M_- A^\dagger\,.$$

Note: For $s = 1/2$ this is identical to the definition of a supersymmetric Dirac Hamiltonian.

Properties:

- Consider
$$H_s^2 \;=\; \begin{pmatrix} (-1)^{2s+1}AA^\dagger + M_+^2 & 0 \\ 0 & (-1)^{2s+1}A^\dagger A + M_-^2 \end{pmatrix}$$
Let $m > 0$ be an arbitrary mass-like parameter and define
$$H_+ := \frac{1}{2mc^2} AA^\dagger \geq 0\,, \qquad H_- := \frac{1}{2mc^2} A^\dagger A \geq 0\,,$$
Define supercharges by
$$Q := \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q^\dagger = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}$$
and the SUSY Hamiltonian by
$$H_{SUSY} := \frac{(-1)^{2s+1}}{2mc^2} \left(H_s^2 - \mathcal{M}^2\right) = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$
results in a $N = 2$ SUSY QM system with $W = \beta$
$$H_{SUSY} = \{Q, Q^\dagger\}\,, \qquad \{Q, W\} = 0\,, \qquad Q^2 = 0 = (Q^\dagger)^2\,.$$
- As for the Dirac case one can show that for such supersymmetric H_s exists a Foldy–Wouthuysen transformation U which diagonalises H_s
$$H_s^{FW} := U H_s U^\dagger = \begin{pmatrix} \sqrt{M_+^2 + (-1)^{2s+1}AA^\dagger} & 0 \\ 0 & -\sqrt{M_-^2 + (-1)^{2s+1}A^\dagger A} \end{pmatrix} = \beta|H_s|$$
The transformation explicitly reads (without proof)
$$U = \frac{|H_s| + \beta H_s}{\sqrt{2H_s^2 + 2\mathcal{M}|H_s|}} = \frac{1 + \beta\,\mathrm{sgn}\,H_s}{\sqrt{2 + \{\mathrm{sgn}\,H_s, \beta\}}}$$

- Due to the SUSY requirement we have $[H_\pm, M_\pm] = 0$ and we can introduce a joint set of eigenfunctions $\phi_\varepsilon^\pm$, this is a $(2s+1)$-spinor, with

$$H_\pm \phi_\varepsilon^\pm = \varepsilon \phi_\varepsilon^\pm \,, \qquad M_\pm \phi_\varepsilon^\pm = m_\varepsilon c^2 \phi_\varepsilon^\pm \,, \qquad \varepsilon \geq 0\,.$$

  Hence the spectral properties of H_s^{FM} can be expressed in terms of $\phi_\varepsilon^\pm$ and ε

$$E_\pm = \pm\sqrt{m_\varepsilon^2 c^4 + (-1)^{2s+1} 2mc^2\varepsilon}\,, \qquad \widetilde{\Psi}_\varepsilon^+ = \begin{pmatrix} \phi_\varepsilon^+ \\ 0 \end{pmatrix}, \qquad \widetilde{\Psi}_\varepsilon^- = \begin{pmatrix} 0 \\ \phi_\varepsilon^- \end{pmatrix},$$

  The SUSY transformations explicitly read for $\varepsilon > 0$

$$\phi_\varepsilon^+ = \frac{1}{\sqrt{2mc^2\varepsilon}}\, A\, \phi_\varepsilon^- \,, \qquad \phi_\varepsilon^- = \frac{1}{\sqrt{2mc^2\varepsilon}}\, A^\dagger\, \phi_\varepsilon^+ \,.$$

  The spectrum is symmetric about zerowith possible exception at m_0c^2 and/or $-m_0c^2$ in case of unbroken SUSY with $\ker A^\dagger$ and/or $\ker A$ being not empty, respectively.

Examples

We consider spin-s particles with mass $m > 0$ and charge e in external magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$.

- The Klein-Gordon Hamiltonian $s = 0$:
  The non-relativistic quantum dynamics is provided by the Landau Hamiltonian

$$H_L := \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 \qquad \text{acting on} \qquad L^2(\mathbb{R}^3)$$

  In 1958 Feshbach and Villars showed that the relativistic Klein-Gordon Hamiltonian is given by

$$H_0 = \begin{pmatrix} mc^2 + H_L & H_L \\ -H_L & -(mc^2 + H_L) \end{pmatrix} \qquad \text{acting on} \qquad L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$$

  Obviously we may identify

$$M_\pm = H_L + mc^2 \,, \qquad A = H_L = A^\dagger \qquad \Longrightarrow \qquad [M_\pm, A] = 0$$

  Hence it is a supersymmetric spin-zero Hamiltonian with

$$H_\pm = \frac{1}{2mc^2}\, H_L^2$$

  The diagonlised FW Hamiltonian reads

$$H_0^{FW} = \begin{pmatrix} \sqrt{(mc^2 + H_L)^2 - H_L^2} & 0 \\ 0 & -\sqrt{(mc^2 + H_L)^2 - H_L^2} \end{pmatrix} = \beta mc^2 \sqrt{1 + \frac{2H_L}{mc^2}}$$

  For a constant magnetic field $\vec{B} = B\vec{e}_z$ the eigenvalues of H_L are the well-know Landau levels

$$\epsilon := \hbar\omega_c \left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m}\,, \qquad n \in \mathbb{N}_0\,, \qquad k_z \in \mathbb{R}\,, \qquad \omega_c := \frac{|eB|}{mc}\,.$$

  Note, the eigenvalues of $H_\pm = \frac{H_L^2}{2mc^2}$ are given by $\varepsilon = \frac{\epsilon^2}{2mc^2} > 0$ and SUSY is broken. The eigenvalues of $M_\pm$ are given by $m_\varepsilon = \epsilon + mc^2 = mc^2\left(1 + \sqrt{\frac{2\varepsilon}{mc^2}}\right)$

- The Dirac Hamiltonian $s = 1/2$:
  The non-relativistic quantum dynamics is provided by the Pauli Hamiltonian with $g = 2$
  $$H_P := \frac{1}{2m}\left[\vec{\sigma}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right)\right]^2 \qquad \text{acting on} \qquad L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$$
  The relativistic Dirac Hamiltonian is given by
  $$H_{1/2} = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right) \\ c\vec{\sigma}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right) & -mc^2 \end{pmatrix} \qquad \text{acting on} \qquad L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$$
  We already know that it is supersymmetric with $M_\pm = mc^2$ and $A = c\vec{\sigma}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right)$.
  The partner Hamiltonians are given by
  $$H_\pm = \frac{1}{2mc^2}\,A^2 = H_P$$
  The diagonlised FW Hamiltonian reads
  $$H^{FW}_{1/2} = \begin{pmatrix} \sqrt{m^2c^4+2mc^2H_P} & 0 \\ 0 & -\sqrt{m^2c^4+2mc^2H_P} \end{pmatrix} = \beta mc^2\sqrt{1+\frac{2H_P}{mc^2}}$$
  For a constant magnetic field $\vec{B} = B\vec{e}_z$ the eigenvalues of H_P are shifted Landau levels
  $$\varepsilon := \hbar\omega_c\left(n+\tfrac{1}{2}+s_z\right)+\tfrac{\hbar^2k_z^2}{2m}\,, \qquad n\in\mathbb{N}_0\,, \qquad k_z\in\mathbb{R}\,, \qquad s_z = \pm\tfrac{1}{2}\,.$$
  SUSY is unbroken here due to the shift!

- The vector boson Hamiltonian $s = 1$:
  The non-relativistic quantum dynamics is provided by the "vector" Hamiltonian for $g = 2$
  $$H_V := \frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{mc}(\vec{S}\cdot\vec{B}) \qquad \text{acting on} \qquad L^2(\mathbb{R}^3)\otimes\mathbb{C}^3$$
  Here $\vec{S} = (S_1,S_2,S_3)^T$ are the spin-1 matrices obeying $[S_i,S_j] = \mathrm{i}\varepsilon_{ijk}S_k$,
  $$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
  The relativistic Hamiltonian describing a vector boson with $g = 2$ is given by
  $$H_1 = \begin{pmatrix} mc^2+H_V & \frac{\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2}{2m} - \frac{\left((\vec{p}-\frac{e}{c}\vec{A})\cdot\vec{S}\right)^2}{m} \\ -\frac{\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2}{2m} + \frac{\left((\vec{p}-\frac{e}{c}\vec{A})\cdot\vec{S}\right)^2}{m} & -(mc^2+H_V) \end{pmatrix} \qquad \text{on} \qquad L^2(\mathbb{R}^3)\otimes\mathbb{C}^6$$
  With $M_\pm = mc^2+H_V$ and $A = \frac{\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2}{2m} - \frac{\left((\vec{p}-\frac{e}{c}\vec{A})\cdot\vec{S}\right)^2}{m} = A^\dagger$ one can show that, for a **constant** magnetic field $[M_\pm, A] = 0$, leading to a supersymmetric relativistic spin-1 Hamiltonian. In addition one may show that $H_V^2 = A^2$.
  The diagonalised FW Hamiltonian then reads
  $$H^{FW}_1 = \begin{pmatrix} \sqrt{(mc^2+H_V)^2-H_V^2} & 0 \\ 0 & -\sqrt{(mc^2+H_V)^2-H_V^2} \end{pmatrix} = \beta mc^2\sqrt{1+\frac{2H_V}{mc^2}}$$
  The eigenvalues of $H_V = H_L - \mathrm{sgn}(eB)\,\hbar\omega_c\,S_3$ are again given by the Landau levels
  $$\epsilon := \hbar\omega_c\left(n+\tfrac{1}{2}+s_z\right)+\tfrac{\hbar^2k_z^2}{2m}\,, \qquad n\in\mathbb{N}_0\,, \qquad k_z\in\mathbb{R}\,, \qquad s_z\in\{-1,0,1\}\,.$$

The partner Hamiltonians $H_\pm = \frac{1}{2mc^2}\, H_V^2$ have the eigenvalues $\varepsilon = \frac{\epsilon^2}{2mc^2}$.
The eigenvalues of $M_\pm$ are given by $m_\varepsilon = \epsilon + mc^2 = mc^2\left(1 + \sqrt{\frac{2\varepsilon}{mc^2}}\right)$.

Note that $\varepsilon = 0$ when $\epsilon = 0$, which is the case for $n = 0$, $s_z = -1$ and $k_z = \pm 1/\lambda_L$.
$\lambda_L := \sqrt{\hbar/m\omega_c} = \sqrt{\hbar c/|eB|}$ is the Larmor wavelength.
Hence SUSY is unbroken, but $\Delta = 0$ as $H_+ = H_-$.

The corresponding eigenvalues of H_1 are then given by

$$E_\pm = \pm\sqrt{m^2c^4 + \hbar^2c^2k_z^2 + 2mc^2\hbar\omega_c(n + 1/2 + s_z)}$$

Note: For $k_z = 0$, $n = 0$ and $s_z = -1$, the above eigenvalue would become complex if $|B| > m^2c^3/|e|\hbar$. Such large magnetic fields would imply $\lambda_L < \lambdabar_C := \hbar/mc$. That is, the Larmor wavelength is small than the reduced Compton wavelength.

Let's confine a particle to such a small area $\Delta x \sim \lambdabar_C$.
Then uncertainty relation implies $\Delta p \sim \hbar/\Delta x = mc$. At such large energies a single particle description is no longer appropriate. In other words for such large magnetic fields a description via quantum field theory must be applied.

For details see GJ, Symmetry 12 (2020) 1590 (14pp)

Summary Section 8

Supersymmetric Dirac Hamiltonians are of the form

$$H_D = \begin{pmatrix} M_+ & A \\ A^\dagger & -M_- \end{pmatrix} \qquad \text{with} \qquad M_+ A = A M_- \,, \quad M_- A^\dagger = A^\dagger M_+ \,.$$

The $N = 2$ SUSY is explicated via ($m > 0$ is a free parameter with dimension of a mass)

$$Q = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix},$$

$$H_{SUSY} = \{Q, Q^\dagger\} = \frac{1}{2mc^2} \left(H_D^2 - \mathcal{M}^2\right) = \frac{1}{2mc^2} \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad W = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note

$$H_+ = \frac{AA^\dagger}{2mc^2} \,, \quad H_- = \frac{A^\dagger A}{2mc^2} \,, \quad [M_+, H_+] = 0 = [M_-, H_-]$$

Supersymmetric Dirac Hamiltonians can always be diagonalised via a FW transformation

$$H_D^{FW} = U H_D U^\dagger = \beta |H_D| = \begin{pmatrix} \sqrt{M_+^2 + 2mc^2 H_+} & 0 \\ 0 & -\sqrt{M_-^2 + 2mc^2 H_-} \end{pmatrix}.$$

The spectral properties of H_D are fully determined by those of the non-relativistic Pauli-like partner Hamiltonians $H_\pm$ and the often trivial mass operators $M_\pm$.

☺