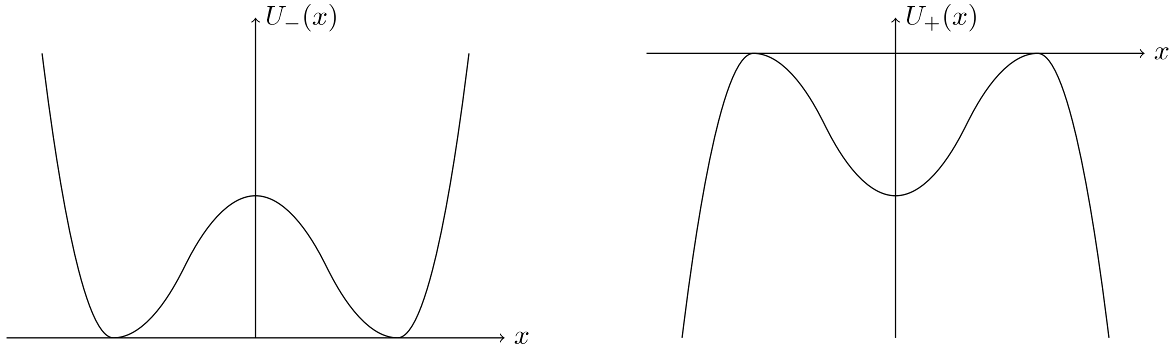
6.3 Supersymmetry of the FP equation

Consider pair of drift potentials $U_\pm$ defined via forces $U'_\pm = F_\pm := \mp\Phi(x)$ or

$$U_\pm(x) := \mp \int_0^x \mathrm{d}z\, \Phi(z) = -U_\mp(x).$$



stable meta stable

FP equation:

$$\partial_t\, m_t^\pm(x,x_0) = \frac{D}{2}\,\partial_x^2\, m_t^\pm(x,x_0) \mp \partial_x\, \Phi(x) m_t^\pm(x,x_0) \qquad \text{with} \qquad m_0^\pm(x,x_0) = \delta(x-x_0)$$

Ansatz:

$$m_t^\pm(x,x_0) =: \exp\left\{-\frac{1}{D}\left[U_\pm(x) - U_\pm(x_0)\right]\right\} K_t^\pm(x,x_0) \qquad \text{with} \qquad K_0^\pm(x,x_0) = \delta(x-x_0)$$

leads to

$$\begin{aligned}
\partial_x\, m_t^\pm(x,x_0) &= \mathrm{e}^{-[\cdots]/D}\left(-\frac{1}{D}U'_\pm(x)K_t^\pm(x,x_0) + \partial_x K_t^\pm(x,x_0)\right)\\
&= \mathrm{e}^{-[\cdots]/D}\left(\partial_x K_t^\pm(x,x_0) \pm \frac{1}{D}\Phi(x)K_t^\pm(x,x_0)\right)\\
\partial_x^2\, m_t^\pm(x,x_0) &= \mathrm{e}^{-[\cdots]/D}\left(\partial_x^2 K_t^\pm(\cdot) \pm \frac{2}{D}\Phi(x)K_t^\pm(\cdot) + \frac{\Phi^2(x)}{D^2}K_t^\pm(\cdot) \pm \frac{\Phi'(x)}{2}K_t^\pm(\cdot)\right)
\end{aligned}$$

In FP equation multiplied by D

$$-D\partial_t K_t^\pm(x,x_0) = \left(-\frac{D^2}{2}\partial_x^2 + \frac{1}{2}\Phi^2(x) \pm \frac{D}{2}\Phi'(x)\right) K_t^\pm(x,x_0)$$

Time-dependent imaginary-time Schrödinger eq. for pair of Hamiltonians

$$H_\pm^{FP} := -\frac{D^2}{2}\partial_x^2 + \frac{1}{2}\Phi^2(x) \pm \frac{D}{2}\Phi'(x)$$

One-to-one correspondence with partner Hamiltonians of Witten model

Witten Model	$\Longleftrightarrow$	Pair of FP
$H_\pm \geq 0$		$H_\pm^{FP} \geq 0$
$\mathrm{i}t$		t
$\hbar$		D
m		1
Φ		$\frac{1}{\sqrt{2}}\Phi$

Solution: Is given by the Euclidean time evolution operator (density matrix)

$$K_t^{\pm}(x, x_0) = \langle x|\mathrm{e}^{-tH_{\pm}^{FP}/D}|x_0\rangle$$

Assume: Purely discrete spectrum for simplicity, that is,

$$H_{\pm}^{FP}|\phi_n^{\pm}\rangle = \lambda_n^{\pm}|\phi_n^{\pm}\rangle\,, \qquad n \in \mathbb{N}_0,$$

Then

$$m_t^{\pm}(x, x_0) = \exp\left\{-\frac{1}{D}\left[U_{\pm}(x) - U_{\pm}(x_0)\right]\right\}\sum_{n=0}^{\infty}\exp\left\{-\frac{1}{D}\,t\lambda_n^{\pm}\right\}\phi_n^{\pm}(x)\phi_n^{\pm *}(x_0).$$

Remarks:

- $\lambda_n^{\pm} \geq 0$ are the decay rates for $U_{\pm}$
- $\phi_n^{\pm}(x)$ are the corresponding decay modes

Stationary distribution: $\Longleftrightarrow \lambda_0^{\pm} = 0 \Longleftrightarrow$ unbroken SUSY with $\Delta = \mp 1$

$$P_{\mathrm{st}}^{\pm}(x) = \lim_{t\to\infty} m_t^{\pm}(x, x_0) = \exp\left\{-\frac{1}{D}\left[U_{\pm}(x) - U_{\pm}(x_0)\right]\right\}\phi_0^{\pm}(x)\phi_0^{\pm *}(x_0)$$

Recall

$$\phi_0^{\pm}(x) = \mathcal{N}\exp\left\{\pm\frac{\sqrt{2m}}{\hbar}\int \mathrm{d}x\,\Phi(x)\right\} = \mathcal{N}\exp\left\{\pm\frac{1}{D}\int \mathrm{d}x\,\Phi(x)\right\} = \mathcal{N}\exp\left\{-\frac{1}{D}U_{\pm}(x)\right\}$$

Hence

$$P_{\mathrm{st}}^{\pm}(x) = |\phi_0^{\pm}(x)|^2$$

Is normalisable in case of unbroken SUSY, i.e. $U_{\pm}(x) \to \infty$ fast enough.
Note, in the case of unbroken SUSY only one of below cases exist

$\Delta = +1$: $P_{\mathrm{st}}^{-}(x) = |\phi_0^{-}(x)|^2$ exists, U_- is stable, U_+ is unstable

$\Delta = -1$: $P_{\mathrm{st}}^{+}(x) = |\phi_0^{+}(x)|^2$ exists, U_+ is stable, U_- is unstable

Obviously "$P_{\mathrm{st}}^{-}(x) = \dfrac{1}{P_{\mathrm{st}}^{+}(x)}$"

Factorisation:
Recall $A = \frac{\hbar}{\sqrt{2m}}\partial_x + \Phi(x) \Longrightarrow A = \frac{1}{\sqrt{2}}(D\partial_x + \Phi(x))$, $A^{\dagger} = \frac{1}{\sqrt{2}}(-D\partial_x + \Phi(x))$

$\Longrightarrow \qquad H_+^{FP} = AA^{\dagger} \geq 0 \qquad H_-^{FP} = A^{\dagger}A \geq 0$

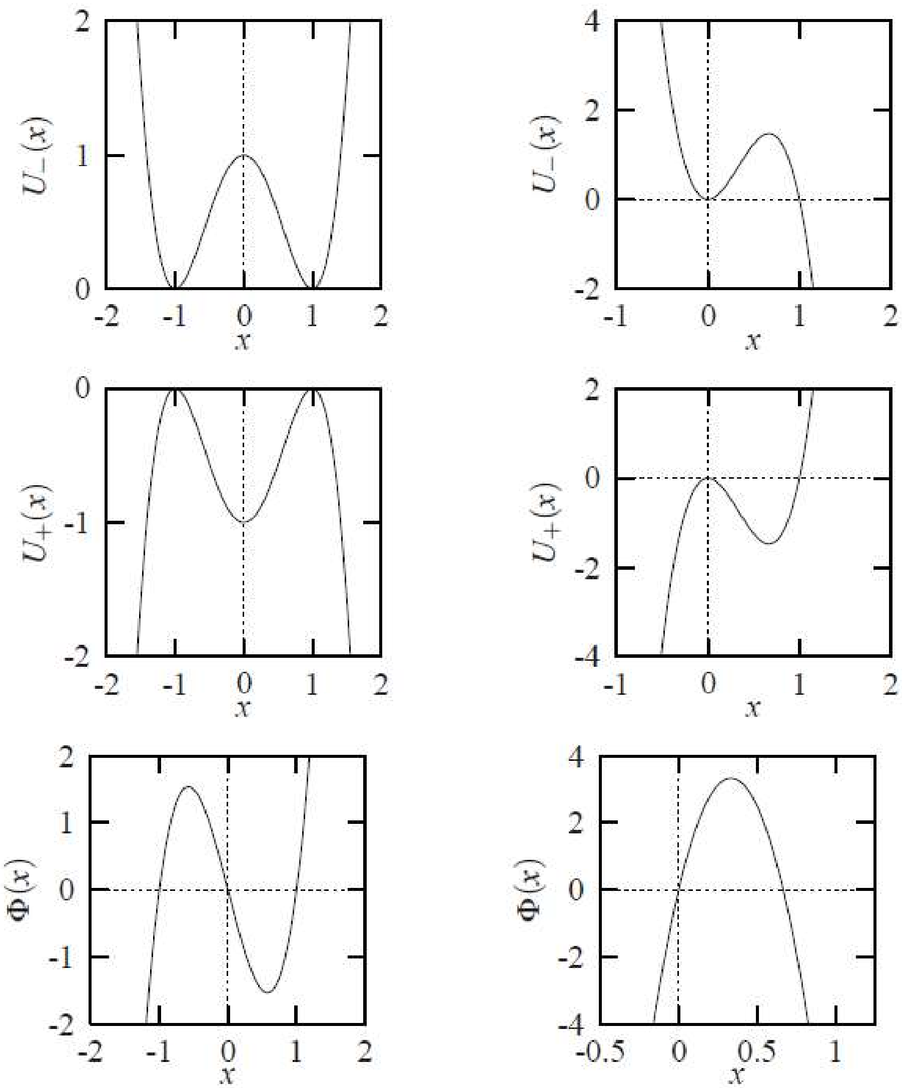


Figure 7.1: Typical drift potentials U_-, inverted drift potentials $U_+ = -U_-$ and drift coefficients $\Phi = \mp U'_\pm$ for good SUSY (*left row*) and broken SUSY (*right row*).

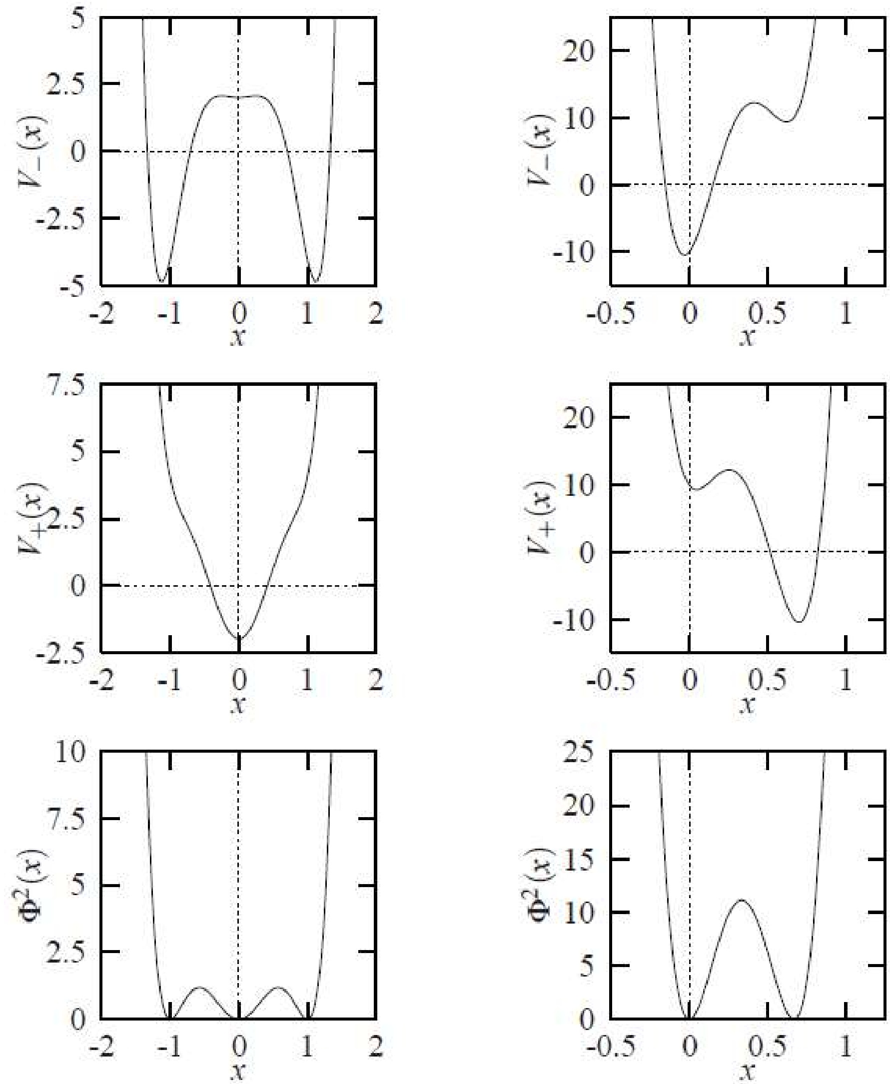


Figure 7.2: The partner potentials $V_\pm$ and Φ^2 for the drift potentials shown in Figure 7.1. Again the left row corresponds to the good-SUSY and the right to the broken-SUSY case. The diffusion constant D has been set to unity.

6.3.1 Implications of unbroken SUSY

We use convention $\Delta = +1$, hence U_- is the stable potential and U_+ is unstable.

- **Stationary distribution**: $\lambda_0^- = 0$

$$P_{\rm st}(x) = |\mathcal{N}|^2 \mathrm{e}^{-\frac{2}{D} U_-(x)} = |\phi_0^-(x)|^2$$

- **Decay rates:**

$$\lambda_n := \lambda_n^- = \lambda_{n-1}^+ > 0\,, \qquad n = 1, 2, 3, \dots$$

  Note: U_+ and $U_- = -U_+$ have identical decay rates!

- **SUSY transformations:** Relation between decay modes

$$\begin{aligned}
\phi_{n-1}^+(x) &= \frac{1}{\sqrt{2\lambda_n}} \left(D\,\frac{\partial}{\partial x} + \Phi(x) \right) \phi_n^-(x),\\
\phi_n^-(x) &= \frac{1}{\sqrt{2\lambda_n}} \left(-D\,\frac{\partial}{\partial x} + \Phi(x) \right) \phi_{n-1}^+(x),
\end{aligned}$$

- **Transition probability density:** Spectral representation

$$\boxed{\begin{aligned}
m_t^-(x, x_0) = &\quad |\phi_0^-(x)|^2 + \frac{\phi_0^-(x)}{\phi_0^-(x_0)} \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t/D} \phi_n^-(x) \phi_n^{-*}(x_0),\\
m_t^+(x, x_0) = &\quad \frac{\phi_0^-(x_0)}{\phi_0^-(x)} \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t/D} \phi_{n-1}^+(x) \phi_{n-1}^{+*}(x_0).
\end{aligned}}$$

$\tau := D/\lambda_1$: time scale for decay of U_+ = time scale of U_- to reach $P_{\rm st}$.

6.3.2 Implications of broken SUSY

- **Decay rates:**

$$\lambda_n := \lambda_n^- = \lambda_n^+ > 0\,, \qquad n = 0, 1, 2, 3, \dots$$

  As before: U_+ and $U_- = -U_+$ have identical decay rates! No stationary distribution.

- **SUSY transformations:**

$$\phi_n^\pm(x) = \frac{1}{\sqrt{2\lambda_n}} \left(\pm D\,\frac{\partial}{\partial x} + \Phi(x) \right) \phi_n^\mp(x).$$

- **Transition probability density:** Spectral representation

$$\boxed{m_t^\pm(x, x_0) = \exp\left\{ \pm \frac{1}{D} \left[U_-(x) - U_-(x_0) \right] \right\} \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_n t/D} \phi_n^\pm(x) \phi_n^{\pm *}(x_0),}$$

  Note: $\exp\left\{\pm\frac{1}{D}\left[U_-(x) - U_-(x_0)\right]\right\} = \exp\left\{-\frac{1}{D}\left[U_\pm(x) - U_\pm(x_0)\right]\right\}$

6.3.3 Some examples

$$\left.\begin{aligned}
\Phi_1(x) &= a\,\mathrm{sgn}\,x\\
\Phi_2(x) &= a\tanh x\\
\Phi_3(x) &= a - \mathrm{e}^{-x}
\end{aligned}\right\} \quad \text{for} \quad a > 0 \quad \text{unbroken SUSY (Homework)}$$

Case 3:

Zero mode: $\phi_0^-(x) = \mathcal{N}\exp\left\{-\int \mathrm{d}x\,\Phi_3(x)\right\} = \mathcal{N}\exp\left\{-ax - \mathrm{e}^{-x}\right\}$

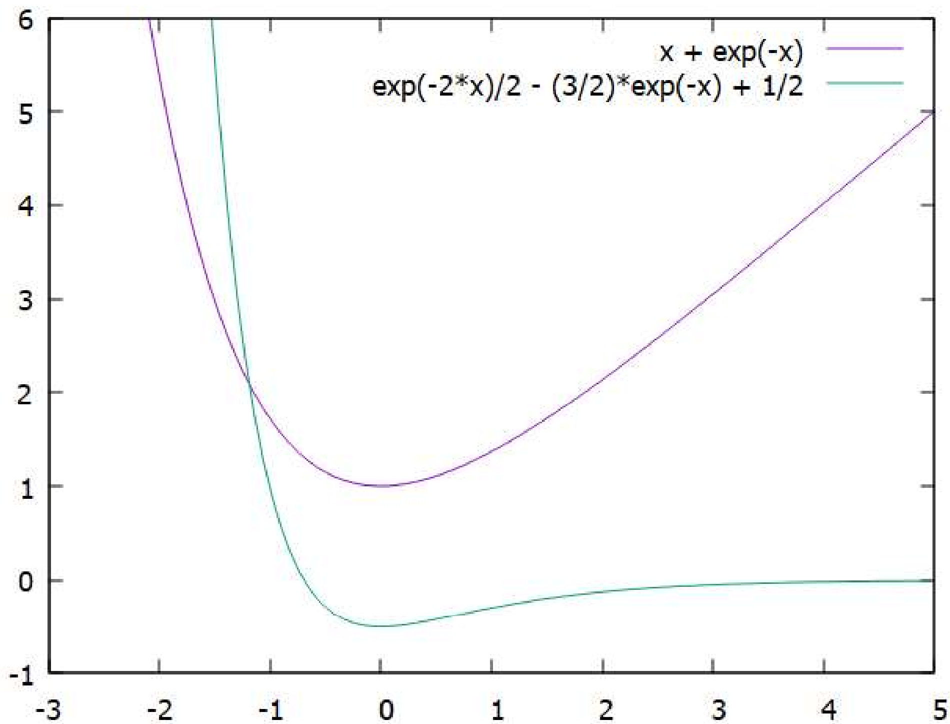
Stationary distribution: $P_{\mathrm{st}}(x) = |\phi_0^-(x)|^2 = \mathcal{N}^2\exp\left\{-2ax - 2\mathrm{e}^{-x}\right\}$

Drift potential: $U_-(x) = \int \mathrm{d}x\,\Phi_3(x) = ax + \mathrm{e}^{-x}$

Partner potentials: $V_\pm(a,x) = \frac{1}{2}\Phi_3^2(x) \pm \frac{1}{2}\Phi_3'(x) = \frac{1}{2}\mathrm{e}^{-2x} - (a \mp \frac{1}{2})\mathrm{e}^{-x} + \frac{1}{2}a^2$

Note: $V_+(a,x) = V_-(a-1,x) + a - \frac{1}{2}$ (shape-inv. Morse potential)

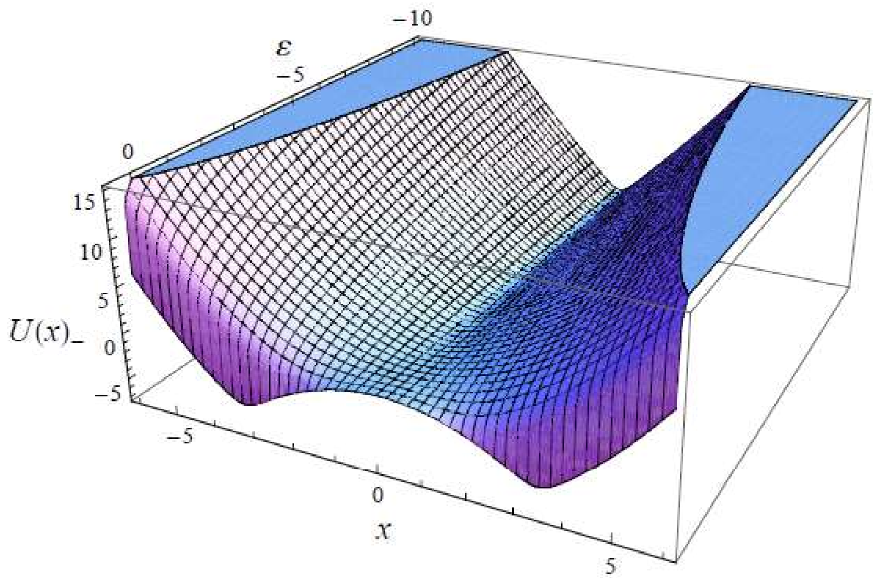
Obviously $\lambda_1 = a - \frac{1}{2}$ if $a > 1$ or $\lambda_1 = \frac{a^2}{2}$ if $0 < a < 1$ (V_- has only 1 bound state).



Additional Homework: Discuss $\Phi(x) = x$

As for the Witten model one can construct conditionally exactly solvable drift potentials (see Book)

Family of stable drift potentials related to the harmonic oscillator



Summary of Section 6

- SUSY naturally appears in Fokker-Planck equation.
- Also for the Langevin equation (see the book section 7.3)
- Diffusion in drift potential U_- and in its inverted potential $U_+ = -U_-$ are closely related.
- For broken SUSY both have same decay rates.
- For unboken SUSY (U_- stable) equilibrium distribution is given by the SUSY ground state, relaxation times into equilibrium are also the decay rates for U_+.
- "Supersymmetric theory of stochastic dynamics" first introduced (1979-1982) by G. Parisi (Nobel price 2021) and N. Sourlas.

7 Supersymmetry in the Pauli-Hamiltonian

7.1 $N = 1$ SUSY of Pauli-Hamiltonian in 3 Dimensions

Spin $\frac{1}{2}$ particle with mass $m > 0$ and charge e ($e < 0$ for electron) in external el.-magn. field characterised by

Vector potential: $\vec{A}(\vec{r}, t)$
Scalar potential: $\phi(\vec{r}, t)$
Hilbert space: $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$

Phenomenological Pauli-Hamiltonian

$$H := \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}(\vec{r}, t)\right)^2 - \vec{\mu}_S \cdot \vec{B}(\vec{r}, t) + e\phi(\vec{r}, t)$$

Magnetic field: $\vec{B}(\vec{r}, t) = \vec{\nabla} \times \vec{A}(\vec{r}, t)$
Electric field: $\vec{E}(\vec{r}, t) = \vec{\nabla}\phi(\vec{r}, t)$
Gauge transformations:

$$\tilde{\phi}(\vec{r}, t) = \phi(\vec{r}, t) - \frac{1}{c}\dot{\chi}(\vec{r}, t)\,, \quad \tilde{\vec{A}}(\vec{r}, t) = \vec{A}(\vec{r}, t) + \vec{\nabla}\chi(\vec{r}, t)\,, \quad \tilde{\psi}(\vec{r}, t) = \mathrm{e}^{\frac{\mathrm{i}e}{\hbar c}\chi(\vec{r}, t)}\psi(\vec{r}, t)$$

Spin: $\vec{S} := \frac{\hbar}{2}\vec{\sigma}$ with Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \vec{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}.$$

Magnetic moment: $\vec{\mu}_S := g\frac{e}{2mc}\vec{S} = g\frac{e\hbar}{4mc}\vec{\sigma} = \frac{g}{2}\,\mathrm{sgn}\,e\,\mu_B\,\vec{\sigma}$
Bohr magenton: $\mu_B := \frac{|e|\hbar}{2mc}$ g: Landé g-factor interaction term $H_S := -\vec{\mu}_S \cdot \vec{B}$
For electrons $e < 0$:

non-relativistic SUSY:	$g = 2$
relativistic Dirac SUSY theory:	$g = 2$
standard model theory:	$g = 2.002\,319\,304\,363\,22(46)$
experiment:	$g = 2.002\,319\,304\,363\,56(35)$

We know from Tutorial 1: From now on $\phi = 0$ and $\dot{\vec{A}} = 0$

$N = 1$ SUSY with $Q = \frac{1}{\sqrt{4m}}\left(\vec{P} - \frac{e}{c}\vec{A}\right) \cdot \vec{\sigma} = Q^\dagger$

No Witten operator but helicity operator $\Lambda = \frac{m\vec{V}\cdot\vec{\sigma}}{\sqrt{2mH}} = \mathrm{sgn}\,Q$

Velocity operator $\vec{V} = \dot{\vec{r}} = \frac{\mathrm{i}}{\hbar}[H, \vec{r}] = \frac{1}{m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)$

SUSY Pauli-Hamiltonian:

$$\boxed{H = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{2mc}\vec{\sigma} \cdot \vec{B}}$$

Homework:
Show $\Lambda^\dagger = \Lambda$, $\Lambda^2 = 1$, $[\Lambda, H] = 0$, $Q = \mathrm{sgn}\,Q\,|Q| = \sqrt{\frac{H}{2}}\,\Lambda$

7.2 $N = 2$ SUSY of Pauli-Hamiltonian in 2 Dimensions

Vector potential: $\vec{A}(x_1, x_2) = \begin{pmatrix} a_1(x_1, x_2) \\ a_2(x_1, x_2) \end{pmatrix}$

Magnetic field: $\vec{B}(x_1, x_2) = B(x_1, x_2)\vec{e}_3\,,$ $\qquad B(x_1, x_2) = \partial_1 a_2(x_1, x_2) - \partial_2 a_1(x_1, x_2)$

Hilbert space: $\mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2 = \mathcal{H}^+ \oplus \mathcal{H}^-$

Witten operator: $W = \sigma_3$, $\qquad \mathcal{H}^\pm = L^2(\mathbb{R}^2)$ spin up/down subspace

Supercharge:
$$Q := A \otimes \sigma^+ = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad W + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
with
$$A := \frac{1}{\sqrt{2m}} \left[\left(P_1 - \frac{e}{c} a_1\right) \mp \mathrm{i} \left(P_2 - \frac{e}{c} a_2\right) \right]$$
$\Longrightarrow$ $N = 2$ SUSY as $Q \neq Q^\dagger$

Hamiltonian: $H := \{Q, Q^\dagger\} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix}$

Calculation:
$$\begin{aligned} AA^\dagger &= \tfrac{1}{2m} \left[\left(P_1 - \tfrac{e}{c} a_1\right) \mp \mathrm{i} \left(P_2 - \tfrac{e}{c} a_2\right) \right] \left[\left(P_1 - \tfrac{e}{c} a_1\right) \pm \mathrm{i} \left(P_2 - \tfrac{e}{c} a_2\right) \right] \\ &= \tfrac{1}{2m} \left[\left(P_1 - \tfrac{e}{c} a_1\right)^2 + \left(P_2 - \tfrac{e}{c} a_2\right)^2 \mp \mathrm{i} \left[\left(P_2 - \tfrac{e}{c} a_2\right)\left(P_1 - \tfrac{e}{c} a_1\right) - \left(P_1 - \tfrac{e}{c} a_1\right)\left(P_2 - \tfrac{e}{c} a_2\right) \right] \right] \\ &= \tfrac{1}{2m} \left(\vec{P} - \tfrac{e}{c}\vec{A}\right)^2 \mp \mathrm{i} \tfrac{e}{2mc} \left([P_1, a_2] + [a_1, P_2]\right) \\ &= \tfrac{1}{2m} \left(\vec{P} - \tfrac{e}{c}\vec{A}\right)^2 \mp \mathrm{i} \tfrac{e\hbar}{2mc\mathrm{i}} \left(\partial_1 a_2 - \partial_2 a_1\right) \\ &= \tfrac{1}{2m} \left(\vec{P} - \tfrac{e}{c}\vec{A}\right)^2 \mp \tfrac{e\hbar}{2mc} B(x_1, x_2) \end{aligned}$$
Similarly $\qquad A^\dagger A = \tfrac{1}{2m} \left(\vec{P} - \tfrac{e}{c}\vec{A}\right)^2 \pm \tfrac{e\hbar}{2mc} B(x_1, x_2)$

Result:
$$H = \frac{1}{2m} \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 \mp \frac{e\hbar}{2mc} B(x_1, x_2) \sigma_3$$
$\Longrightarrow$ $N = 2$ SUSY of Pauli-Hamiltonian with $g = \pm 2$.

Witten parity eigenstates are eignestates of S_3.

From now on we consider only upper sign $g = +2$ and electrons $e = -|e|$.
$$H = \frac{1}{2m} \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 + \mu_B B(x_1, x_2) \sigma_3$$

Magnetic flux:
$$F := \int_{\mathbb{R}^2} \mathrm{d}x_1 \mathrm{d}x_2 \, B(x_1, x_2)$$
and assume $|F| < \infty$, that is, B is bounded with compact support

Aharonov-Casher theorem: (see Tutorial 14)

- Ground state energy: $E_0 = 0 \qquad \Longrightarrow \qquad$ SUSY unbroken
- Degeneracy of E_0: $\qquad d = \left[\!\left[\frac{|F|}{\Phi_0} \right]\!\right]$
  Here $[\![z]\!] := \max_{n \in \mathbb{N}_0} \{n | n < z\}$, largest integer stricly less then z.
  And $\Phi_0 := 2\pi \frac{\hbar c}{|e|}$ represents the flux quantum.
- All d ground states belong either
  to $\quad \mathcal{H}^-$ for $F > 0$, spin-down states
  or $\quad \mathcal{H}^+$ for $F < 0$, spin-up states
- SUSY implies that all states with $E > 0$ are pairwise ($\uparrow\downarrow$) degenerate due to existing SUSY transformations. Unpaired spins can only exist on the ground state level.
- Witten index:
$$\Delta = \dim \ker A^\dagger A - \dim \ker AA^\dagger = d \,\mathrm{sgn}\, F \approx \frac{F}{\Phi_0}$$
  Topological invariant as details of B are irrelevant and only total flux through $\mathbb{R}^2$ is essential!

7.3 Paramagnetism of non-interacting electrons in 2D

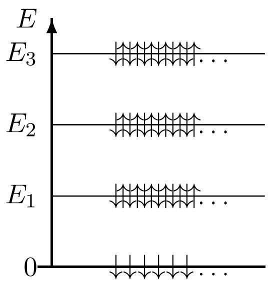
Consider a 2-dim. gas of N non-interacting electrons at $T = 0$

- **Ground state:** Is characterised by integrated density of states

$$\Theta(\varepsilon_F - H)$$

With 1-particle Hamiltonian $\quad H = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 + \mu_B B(x_1, x_2)\sigma_3$
and Fermi energy ε_F defined via $\quad \mathrm{Tr}\,\Theta(\varepsilon_F - H) = N.$
All states up to ε_F are occupied.

- **Typical N-particle ground state:**
Assumption that ε_F is between two Landau levels (case $B = const.$)
All levels either fully occupied or empty



- **Magnetisation:** Recall magnetic moment of single electron $\vec{\mu}_S = -\mu_B \vec{\sigma}$

$$\begin{aligned} M &:= \mu_B\,(N_\downarrow - N_\uparrow) \qquad N_{\uparrow\downarrow} : \text{No. of occupied } \uparrow\downarrow \text{ states} \\ &= -\mu_B \mathrm{Tr}\left[\sigma_3 \Theta(\varepsilon_F - H)\right] \\ &= \mu_B \tilde{\Delta}(\varepsilon_F) \qquad \text{IDOS regulated Witten index} \\ &= \mu_B \Delta \qquad \text{under above assumption} \\ &= \mu_B\, d\,\mathrm{sgn}\, F \approx \mu_B \tfrac{F}{\Phi_0} \qquad \text{topological invariant} \end{aligned}$$

- **Simplifying assumptions:** $B(x_1, x_2) = B > 0$ constant magn. field on finite area $\mathcal{A}$
$\mathcal{A} \subset \mathbb{R}^2$ with $\mathcal{A} := \{(x_1, x_2) \in \mathbb{R}^2 | -\ell/2 \le x_i \le \ell/2, i = 1, 2\} \implies F = B\ell^2 > 0$
magnetisation: $\quad M = \mu_B \dfrac{B\ell^2}{\Phi_0}$
specific magnetisation: $\dfrac{M}{\ell^2} = \mu_B \dfrac{B}{\Phi_0} = \mu_B \dfrac{|e|B}{2\pi\hbar c}$

- **Paramagnetic Susceptibility:** of the 2-dim. electron gas

$$\boxed{\chi := \frac{1}{\ell^2}\frac{\partial M}{\partial B} = \mu_B \frac{|e|}{2\pi\hbar c} = \frac{e^2}{4\pi m c^2}}$$

Remarks:

- Result independent of electron density (ε_f) and magnetic field strength (B)!
- Derivation uses full single-particle Pauli-Hamiltonian

$$H \equiv H^{(2)} = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 + \mu_B B \sigma_3$$

Standard textbook use the free Hamiltonian with spin term

$$H_0 := \frac{1}{2m}\vec{P}^2 + \mu_B B\sigma_3$$

but arrive at same result!!! $\quad \Longrightarrow \quad$ "Topological Invariance"

7.4 Paramagnetism of non-interacting electrons in 3D

Homogeneous magnetic field: $\vec{B} = B\vec{e}_3$ on $\mathcal{A}$ as before
Macroscopic Volume: $V = \ell^2 \ell_3$ $\quad \ell_3$ is extension in x_3-direction
Single particle Hamiltonian:

$$H^{(3)} = \frac{P_3^2}{2m} + H^{(2)}$$

Free motion in x_3-direction but eigenvalues of P_3 are quantised as $\ell_3 < \infty$
periodic boundary conditions allow only certain wavelengths

$$p_3 = \hbar k_3 \qquad \text{with} \qquad k_3 = \frac{2\pi}{\ell_3}\, n\,, \qquad n \in \mathbb{Z}$$

For the non-interaction electron gas all k_3 are occupied where

$$|k_3| < k_F := \frac{\sqrt{2m\varepsilon_f}}{\hbar} \qquad \text{Fermi wave number}$$

Number of occupied k_3: $\qquad 2n_{\max} = \frac{k_F \ell_3}{\pi}\,, \qquad -n_{\max} < n < n_{\max}$
Each eigenvalue k_3 contributes to magnetisation the 2-dim. result

$$M^{(2)} = \mu_B \frac{B\ell^2}{\Phi_0}$$

Total magnetisation:

$$M^{(3)} = 2n_{\max} M^{(2)} = \frac{k_F \ell_3}{\pi} \mu_B \frac{B\ell^2}{\Phi_0}$$

Specific magnetisation:

$$\frac{M^{(3)}}{V} = \frac{k_F B}{\pi} \frac{\mu_B}{\Phi_0} = \frac{e^2}{4\pi^2 mc^2}\, k_F\, B$$

Paramagnetic Susceptibility: Is dimensionless!

$$\chi^{(3)} = \frac{1}{V} \frac{\partial M^{(3)}}{\partial B} = \frac{e^2}{4\pi^2 mc^2}\, k_F = \left(\frac{\alpha}{2\pi}\right)^2 a_0\, k_F$$

Bohr radius: $a_0 := \frac{\hbar^2}{me^2}$
Fine structure constant: $\alpha := \frac{e^2}{\hbar c}$

7.5 The textbook approach

Calculate spectral density of a free particle in a box: $V = L^3$ $\quad$ using $\quad H_0 = \frac{\vec{P}^2}{2m}$

- Eigenfunctions:

$$\psi(\vec{r}) = \left(\tfrac{1}{2\pi}\right)^{3/2} e^{i\vec{k}\cdot\vec{r}}\,, \qquad k_j = \tfrac{2\pi}{L}\, n_j\,, \qquad n_j \in \mathbb{Z}\,, \qquad j = 1,2,3$$

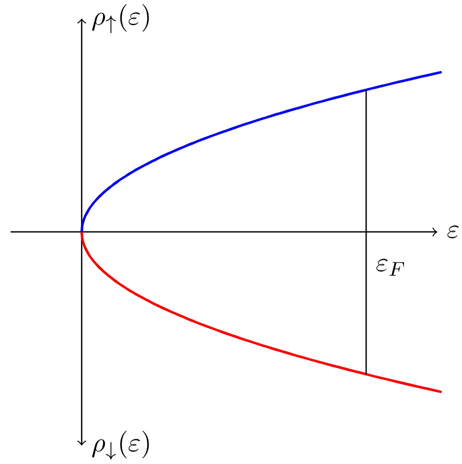
- Volume taken by one state in k-space: $\Omega_0 := \left(\frac{2\pi}{L}\right)^3$
- Volume of sphere in k-space: $\mathrm{d}\Omega = 4\pi k^2 \mathrm{d}k$
  with $\varepsilon(\vec{k}) = \frac{\hbar^2 \vec{k}^2}{2m} \quad \Rightarrow \quad k = |\vec{k}| = \sqrt{\frac{2m\varepsilon}{\hbar^2}} \quad \Rightarrow \quad \mathrm{d}\varepsilon = \frac{\hbar^2 k}{m}\mathrm{d}k$
  Hence $\qquad \mathrm{d}\Omega = 4\pi \frac{2m\varepsilon}{\hbar^2} \frac{m}{\hbar^2 k} \mathrm{d}\varepsilon = 4\pi \frac{m}{\hbar^2} \sqrt{\frac{2m\varepsilon}{\hbar^2}} \mathrm{d}\varepsilon$
- Spectral density: number of states in the sphere

$$D(\varepsilon) := \frac{1}{\Omega_0} \frac{\mathrm{d}\Omega}{\mathrm{d}\varepsilon} = \frac{V}{8\pi^3} 4\pi \frac{m}{\hbar^2} \sqrt{\frac{2m\varepsilon}{\hbar^2}} = \frac{Vm}{2\pi^2 \hbar^2} \sqrt{\frac{2m\varepsilon}{\hbar^2}}$$

- Specific spectral density:

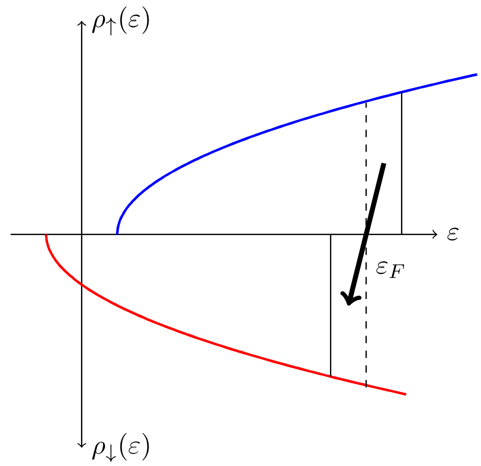
$$\rho(\varepsilon) := \frac{D(\varepsilon)}{V} = \frac{m}{2\pi^2\hbar^2}\sqrt{\frac{2m\varepsilon}{\hbar^2}} = \frac{m}{2\pi^2\hbar^2}\,k$$

Graphical representation for spin-up/-down electrons



Switch on magnetic field: Using free Hamiltonian with spin term only

$$H_0 \qquad \Rightarrow \qquad H_0 = \frac{\vec{P}^2}{2m} + \mu_B B \sigma_3$$



$N_\uparrow \quad \to \quad N_\uparrow - \mu_B B \rho(\varepsilon_F) V$
$N_\downarrow \quad \to \quad N_\downarrow - \mu_B B \rho(\varepsilon_F) V$
Magnetisation: $M^{(3)} = \mu_B(N_\downarrow - N_\uparrow) = 2\mu_B^2 \rho(\varepsilon_F) V B$
Susceptibility: $\chi^{(3)} = 2\mu_B^2\rho(\varepsilon_F) = \frac{e^2\hbar^2}{4m^2c^2}\frac{m}{\pi^2\hbar^2}\,k_F = \frac{e^2}{4\pi^2 mc^2}\,k_F$

Result is identical to the SUSY derivation.
Surprisingly the wrong use of the free Hamiltonian with spin term is sufficient.
The spectral free density actually changes drastically to Landau levels.
Nevertheless the net magnetisation is NOT sensitive to such approximation.

Recall $M = \mu_B\,\Delta$ is related to the Witten index, i.e. a topological invariant.