4 The Darboux Method (1882)

Assumption: Let us assume we have two self-adjoint operators $H_\pm$ and one linear operator A, all acting on common Hilbert space $\mathcal{H}$ obeying the condition

$$\boxed{H_+ A = A H_-} \qquad \Longrightarrow \qquad A^\dagger H_+ = H_- A^\dagger \qquad\qquad (*)$$

Let's further assume the spectral properties of H_+ are known (we assume a purely discrete spectrum for simplicity)

$$H_+|\phi_n^+\rangle = E_n|\phi_n^+\rangle\,, \qquad n = 0, 1, 2, 3, \ldots$$

Then

$$|\phi_n^-\rangle := C_n\, A^\dagger|\phi_n^+\rangle \neq 0$$

is eigenstate of H_- with same eigenvalue E_n.
Obvious as

$$H_-|\phi_n^-\rangle = C_n H_- A^\dagger|\phi_n^+\rangle \overset{(*)}{=} C_n A^\dagger H_+|\phi_n^+\rangle = E_n|\phi_n^-\rangle$$

Remarks:

- States ϕ_n^+ such that $A^\dagger|\phi_n^+\rangle = 0$ do not lead to a $|\phi_n^-\rangle$. Hence, eigenvalues of H_+ associated with states $\phi_n^+ \in \ker A^\dagger$ are in general not eigenvalues of H_-
- With $A|\phi_n^-\rangle \neq 0$ we obtain an eigenstate of H_+. Let $H_-|\phi_n^-\rangle = E_n|\phi_n^-\rangle$ then

$$H_+ A|\phi_n^-\rangle \overset{(*)}{=} A H_-|\phi_n^-\rangle = E_n A|\phi_n^-\rangle$$

- H_- may have additional eigenvalues with eigenstates $\phi_n^- \in \ker A$, i.e. $A|\phi_n^-\rangle = 0$

Conclusion: From spectral properties of H_+ on may conclude those of H_-.
$H_\pm$ are not necessarily Schrödinger operators $\quad\Longrightarrow\quad$ Wide fields of applications

4.1 Modelling Conditionally Exactly Solvable Potentials

Let

$$H_\pm = -\frac{\hbar^2}{2m}\,\partial_x^2 + V_\pm(x) \qquad \text{on} \qquad \mathcal{H} = L^2(\mathbb{R})$$

be two 1-dim. Schrödinger Hamiltonians.

Ansatz for A:

$$A := \sum_{k=0}^{N} f_k(x)\partial_x^k$$

with $f_k : \mathbb{R} \to \mathbb{R}$ being at least twice differentiable.

Insert into defining relation $(*)$ and compare coefficients of same power of ∂_x^k
$\Longrightarrow \qquad$ Solve for the f_k's

Obviously $f_N = const.$ $\qquad$ for convenience we choose $f_N := \hbar/\sqrt{2m}$

4.1.1 The simplest non-trivial case $N = 1$

$$A := \frac{\hbar}{\sqrt{2m}}\,\partial_x + \Phi(x) \qquad \text{with} \qquad \Phi(x) := f_0(x)\,, \qquad f_1 := \hbar/\sqrt{2m}$$

Inserting into $(*)$ results in two coupled equations

$$V_-(x) = V_+(x) - \frac{2\hbar}{\sqrt{2m}}\,\Phi'(x)$$
$$\frac{\hbar}{\sqrt{2m}} V_-'(x) + \Phi(x)V_-(x) = -\frac{\hbar^2}{2m}\Phi''(x) + \Phi(x)V_+(x)$$

Elimination of V_- results in a non-linear Riccati equation

$$\Phi^2(x) + \frac{\hbar}{\sqrt{2m}}\Phi'(x) = V_+(x) - \varepsilon\,.$$

Here $\varepsilon \in \mathbb{R}$ is a constant of integration.

Linearisation with ansatz: $\Phi(x) =: \dfrac{\hbar}{\sqrt{2m}}\dfrac{u'(x)}{u(x)}$

$$\boxed{\left[-\frac{\hbar^2}{2m}\,\partial_x^2 + V_+(x)\right] u(x) = \varepsilon u(x)}$$

Schrödinger-type equation BUT u is NOT required to be square integrable and ε is not necessarily an eigenvalue of H_+. See Tutorial Exercise 8.

Remarks:

- $H_+ = AA^\dagger + \varepsilon$, $\qquad H_- = A^\dagger A + \varepsilon \qquad$ shifted Witten model
- New potential V_- with associated Hamiltonian H_- whose spectral properties are basically known.

$$V_-(x) = \frac{\hbar^2}{m}\left(\frac{u'(x)}{u(x)}\right)^2 - V_+(x) + 2\varepsilon$$

- Condition: $u(x) \neq 0$ for all $x \in \mathbb{R} \qquad \Longrightarrow \qquad$ No singularities!

$$\boxed{\varepsilon \leq E_0 := \min \operatorname{spec} H_+} \qquad \text{Sturm – Liouville Theory}$$

- Consider $\ker A^\dagger$: $A^\dagger|\phi_0^+\rangle = 0 \qquad \Longrightarrow \qquad -\dfrac{\hbar}{\sqrt{2m}}\phi_0^{+\prime}(x) + \Phi(x)\phi_0^+(x) = 0$

  $$\Longrightarrow \qquad \frac{\hbar}{\sqrt{2m}}\frac{\phi_0^{+\prime}(x)}{\phi_0^+(x)} = \Phi(x) = \frac{\hbar}{\sqrt{2m}}\frac{u'(x)}{u(x)} \qquad \Longrightarrow \qquad \phi_0^+(x) = u(x) \text{ nodeless}$$

  $$\Longrightarrow \qquad \varepsilon = E_0$$

  From now on $\varepsilon < E_0 \qquad \Longrightarrow \qquad \ker A^\dagger = \emptyset$.
  Complete spectrum of H_+ belongs to spectrum of H_-. $\qquad \operatorname{spec} H_+ \subset \operatorname{spec} H_-$

- Consider $\ker A$: $A|\phi_\varepsilon^-\rangle = 0 \qquad \Longrightarrow \qquad \phi_\varepsilon^{-\prime}(x) = -\dfrac{u'(x)}{u(x)}\phi_\varepsilon^-(x) \qquad \Longrightarrow$

  $$\boxed{\phi_\varepsilon^-(x) = \frac{C}{u(x)}}$$

  Assume nodeless $u(x) \to \infty$ for $x \to \pm\infty$ such that $\phi_\varepsilon^- \in L^2(\mathbb{R}) \qquad \Longrightarrow$

  $$\operatorname{spec} H_- = \{\varepsilon, E_0, E_1, E_2 \ldots\} = \{\varepsilon\} \cup \operatorname{spec} H_+$$

With $|\phi_n^-\rangle = C_n A^\dagger|\phi_n^+\rangle$ follows $||\phi_n^-||^2 = |C_n|^2\langle\phi_n^+|AA^\dagger|\phi_n^+\rangle = |C_n|^2\langle\phi_n^+|H_+ - \varepsilon|\phi_n^+\rangle$

Hence $|C_n|^2 = \dfrac{1}{E_n - \varepsilon} > 0$.

Summary of results: Given: Known spectral properties $\quad H_+|\phi_n^+\rangle = E_n|\phi_n^+\rangle$

$\Longrightarrow \quad H_-|\phi_n^-\rangle = E_n|\phi_n^-\rangle \quad$ and $\quad H_-|\phi_\varepsilon^-\rangle = \varepsilon|\phi_\varepsilon^-\rangle \quad$ with $\quad \varepsilon < E_0$

with conditionally exactly solvable potential

$$\boxed{V_-(x) = \frac{\hbar^2}{2m}\left(\frac{u'(x)}{u(x)}\right)^2 - V_+(x) + 2\varepsilon}$$

as $\varepsilon < E_0$ and $u(x)$ nodeless where

$$-\frac{\hbar^2}{2m}\,u''(x) + V_+(x)u(x) = \varepsilon u(x)$$

and spectral properties

$$\begin{aligned}
\operatorname{spec} H_- &= \{\varepsilon, E_0, E_1, E_2, \ldots\}\\
\phi_\varepsilon^-(x) &= \frac{C}{u(x)} \in L^2(\mathbb{R})\\
\phi_n^-(x) &= \frac{1}{\sqrt{E_n - \varepsilon}}\left(-\frac{\hbar}{\sqrt{2m}}\phi_n^{+\prime}(x) + \frac{\hbar}{\sqrt{2m}}\frac{u'(x)}{u(x)}\phi_n^+(x)\right)\\
&= \frac{\hbar}{\sqrt{2m(E_n - \varepsilon)}}\left(\frac{u'(x)}{u(x)}\phi_n^+(x) - \phi_n^{+\prime}(x)\right)
\end{aligned}$$

4.2 A family of SUSY partners of the linear harmonic oscillator

For simplicity we set $\hbar = m = \omega = 1$.

$V_+(x) = \frac{1}{2}\,x^2 \quad$ with $\quad E_n = (n + \frac{1}{2})$

Obviously $\varepsilon < \frac{1}{2}$

General solution of Schrödinger-like eq.
(see, e.g., Galindo & Pascual, QMI Springer 1989, p. 143 and appendix A)

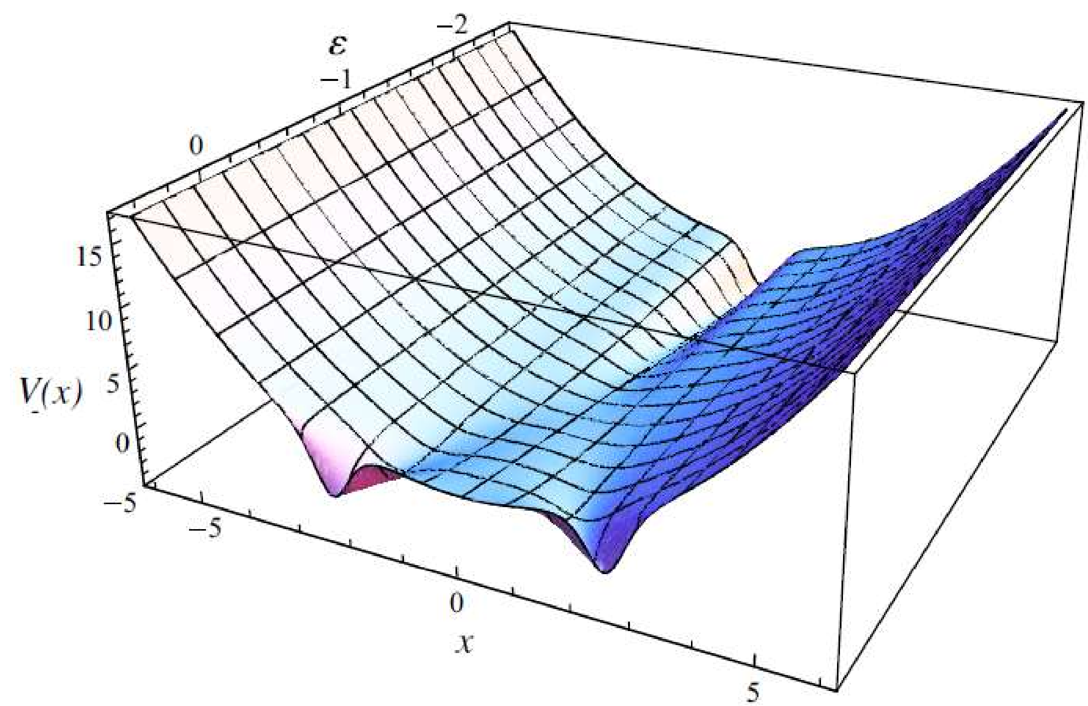
$$u(x) = \mathrm{e}^{-x^2/2}\left[\alpha\ {}_1F_1\left(\frac{1-2\varepsilon}{4}, \frac{1}{2}, x^2\right) + \beta\, x\ {}_1F_1\left(\frac{3-2\varepsilon}{4}, \frac{3}{2}, x^2\right)\right]$$

Confluent hypergeom. function:

$${}_1F_1(a, c, z) \equiv M(a, c, z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!} \qquad \text{with} \qquad (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = a(a{+}1)(a{+}2)\cdots(a{+}n{-}1)$$

For $a = -m$, $m \in \mathbb{N}_0$, this is a polynomial in z of degree m

Symmetric case $\beta = 0$



Remarks:

- Without loss of generality $\alpha = 1$
- $u(x) > 0$ for all $x \in \mathbb{R}$ $\quad\Longrightarrow\quad$ $|\beta| < \beta_c(\varepsilon) := 2\frac{\Gamma(3/4-\varepsilon/2)}{\Gamma(1/4-\varepsilon/2)}$
- $\beta = 0$: $V_-(x) = V_-(-x)$ sym. see figure above
- $\beta \in \mathbb{C}\setminus(]-\infty,-\beta_c]\cup[\beta_c,\infty[)$ allowed $\Longrightarrow$ complex potential with real spectrum
  Area of intensive research in last 20 years

Spectral properties:

$$H_+ : \quad \operatorname{spec} H_+ = \{E_0, E_1, E_2, \ldots\}, \qquad E_n = n + \tfrac{1}{2}$$

$$\phi_n^+(x) = \left(\frac{1}{\sqrt{\pi}2^n n!}\right)^{1/2} \mathrm{e}^{-x^2/2} H_n(x) \qquad \text{Hermite polynomials}$$

$$V_+(x) = \frac{1}{2}\,x^2$$

$$H_- : \quad \operatorname{spec} H_- = \{\varepsilon, E_0, E_1, E_2, \ldots\}, \qquad \varepsilon < \tfrac{1}{2} \qquad \text{arbitrary}$$

$$\phi_\varepsilon^-(x) = \frac{C\,\mathrm{e}^{x^2/2}}{{}_1F_1\left(\frac{1-2\varepsilon}{4}, \frac{1}{2}; x^2\right) + \beta x\,{}_1F_1\left(\frac{3-2\varepsilon}{4}, \frac{3}{2}; x^2\right)}$$

$$\phi_n^-(x) = \frac{\mathrm{e}^{-x^2/2}}{\left[\sqrt{\pi}\,2^{n+1} n!(n+1/2-\varepsilon)\right]^{1/2}} \left[H_{n+1}(x) + \left(\frac{u'(x)}{u(x)} - x\right) H_n(x)\right]$$

$$V_-(x) = \left[\left(\frac{u'(x)}{u(x)}\right)^2 - \frac{1}{2}x^2 + 2\varepsilon\right].$$

Special cases:

- $\varepsilon = -\frac{1}{2}$, $\beta = 0$:

$$u(x) = \mathrm{e}^{-x^2/2}\,{}_1F_1(\tfrac{1}{2}, \tfrac{1}{2}, x^2) = \mathrm{e}^{x^2/2}\,, \qquad \tfrac{u'(x)}{u(x)} = x\,,\ \phi_n^-(x) = \phi_{n+1}^+(x)$$

- $\varepsilon = -\frac{1}{2} - 2k$, $k \in \mathbb{N}_0$, $\beta = 0$:

$$u(x) = \mathrm{e}^{-x^2/2}\,{}_1F_1(k+\tfrac{1}{2}, \tfrac{1}{2}, x^2) = \mathrm{e}_1^{x^2/2}F_1(-k, \tfrac{1}{2}, -x^2) \text{ (Hermite polynomial)}$$

Note: ${}_1F_1(a, c, z) = \mathrm{e}^z\,{}_1F_1(c-a, c, -z))$

$$u(x) = \mathrm{e}^{x^2/2}\underbrace{(-1)^k\frac{k!}{(2k)!}}_{=:1/\alpha}H_{2k}(\mathrm{i}x) = \mathrm{e}^{x^2/2}H_{2k}(\mathrm{i}x)$$

  - $k = 0$: $H_0(\mathrm{i}x) = 1$ previous case
  - $k = 1$: $H_1(\mathrm{i}x) = 4(\mathrm{i}x)^2 - 2 = -4x^2 - 2$ $\Longrightarrow$ Homework
  - k arbitrary:
    $u'(x) = x\mathrm{e}^{x^2/2}H_{2k}(\mathrm{i}x) + \mathrm{i}\mathrm{e}^{x^2/2}H'_{2k}(\mathrm{i}x)\,,\quad H'_{2k}(z) = 2zH_{2k}(z) - H_{2k+1}(z)$ $\Longrightarrow$
    $\frac{u'(x)}{u(x)} = x + \mathrm{i}\frac{H'_{2k}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)} = x + \mathrm{i}2\mathrm{i}x - \mathrm{i}\frac{H_{2k+1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)} = -x - \mathrm{i}\frac{H_{2k+1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)}$

    Rational potential

$$V_-(x) = \frac{x^2}{2} + 2\mathrm{i}x\frac{H_{2k+1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)} - \left(\frac{H_{2k+1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)}\right)^2 - 4k - 1$$

    generates spectrum $\operatorname{spec} H_- = \{-\frac{1}{2} - 2k, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$

For a complete discussion for shape-invariant potentials see GJ & P. Roy, Ann. Phys. 270 (1998) 155
Homework: Find all SUSY partners of the free particle.

Summary of section 4

- Darboux method closely related to SUSY QM but can be extended beyond
- Designing of quantum potentials with known spectral properties. More recently discussion of complex potentials (PT-symmetry)
- The family of harmonic oscillator SUSY partners also inspired new ladder operators obeying a non-linear algebra (see Exercise 9)

5 Classical Fields in $(1+1)$ Dimensions

Consider a scalar field:

$$\phi : \begin{array}{l} \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (x,t) \mapsto \phi(x,t) \end{array}$$

with vanishing variations at infinity, that is,

$$\phi' := \partial_x \phi \to 0 \qquad \text{and} \qquad \dot{\phi} := \partial_t \phi \to 0 \qquad \text{for} \qquad x, t \to \pm\infty.$$

The corresponding Lagrange density is defined as

$$\mathcal{L}(\partial\phi, \phi) := \frac{1}{2}\left(\partial_t \phi\right)^2 - \frac{1}{2}\left(\partial_x \phi\right)^2 - U(\phi) = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - U(\phi)$$

with a real-valued field potential U bounded from below, i.e. $U \geq 0$.
The Euler-Lagrange equation

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

then results in the classical eq. of motion

$$\partial_\mu \partial^\mu \phi + U'(\phi) = 0$$

or more explicitly

$$\boxed{\ddot{\phi} - \phi'' = -\frac{\partial U}{\partial \phi}}.$$

Examples:

- Klein-Gordon: $\qquad U(\phi) = \frac{1}{2}\phi^2$

  $\Longrightarrow \qquad \partial_\mu \partial^\mu \phi + \phi = 0$
  KG equation for rel. scalar field with unit mass
- Sine-Gordon: $\qquad U(\phi) = 1 + \cos\phi$

  $\Longrightarrow \qquad \ddot{\phi} - \phi'' + \sin\phi = 0$
  Instantons / Solitons
- ϕ^4-theory: $\qquad U(\phi) = \frac{1}{2}(1-\phi^2)^2$

  $\Longrightarrow \qquad \ddot{\phi} - \phi'' + 2(1-\phi^2)\phi = 0$
  Phase transitions / Higgs mechanism

Conserved energy functional:

$$E[\phi] := \int_{\mathbb{R}} \mathrm{d}x \, \left[\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\phi'^2 + U(\phi)\right] ,$$

Homework: Show $\dfrac{\mathrm{d}}{\mathrm{d}t} E[\phi] = 0$

Finite energy configurations:
Now in addition we assume that $U(\phi) \to 0$ as $x \to \pm\infty$ (vacuum configuration)
That is, we assume

$$\phi_{\pm} := \lim_{x \to \pm\infty} \phi(x,t) \qquad \text{with} \qquad U(\phi_{\pm}) = 0$$

We further assume translation invariance:

$$\phi(x,t) = \phi_{\text{st}}(x - vt) \qquad\qquad \text{st} = \text{static}$$

These localised solutions are called *solitary waves*

Eq. of motion for a static solution $\phi_{\text{st}}(x)$

$$\begin{aligned} &\phi''_{\text{st}}(x) = U'(\phi_{\text{st}}(x)) \\ \Longrightarrow \qquad &\phi'_{\text{st}}(x)\phi''_{\text{st}}(x) = U'(\phi_{\text{st}}(x))\phi'_{\text{st}}(x) \\ \Longrightarrow \qquad &\tfrac{1}{2}\left[\phi'_{\text{st}}\right]^2 = U(\phi_{\text{st}}) + \varepsilon \end{aligned}$$

Recall $\phi'_{\text{st}} \to 0$ and $U(\phi_{\text{st}}) \to 0$ for $x \to \pm\infty \qquad \Longrightarrow \qquad \varepsilon = 0$

Result:

$$\boxed{\frac{1}{2}{\phi'_{\text{st}}}^2(x) = U(\phi_{\text{st}}(x))}$$

5.1 Stability of static solutions

Consider fluctuations around a static solution

$$\phi(x) = \phi_{\text{st}}(x) + \psi(x)$$

with small fluctuation ψ such that $\psi(x) \to 0 \quad \text{as} \quad x \to \pm\infty$.
That is

$$E[\phi] \approx E[\phi_{\text{st}}] + \delta E[\psi]$$

where (see tutorial)

$$\delta E[\psi] := \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x\, \psi(x)\left[-\partial_x^2 + U''(\phi_s(x))\right]\psi(x)$$

Fluctuation operator:

$$H := -\partial_x^2 + U''(\phi_{\text{st}}(x))$$

Schrödinger-like operator acting on $L^2(\mathbb{R})$.

Assume that we know the eigenmodes, that is,

$$H\psi_n = \mu_n \psi_n\,,$$

then

$$\psi(x) = \sum_n a_n\, \psi_n(x) \qquad \text{with} \qquad a_n := \int_{\mathbb{R}} \mathrm{d}x\, \psi_n^*(x)\psi(x)$$

Hence

$$\delta E[\psi] = \frac{1}{2}\sum_n \mu_n\, |a_n|^2$$

Stability:

$$\delta E[\psi] \geq 0 \qquad \Longleftrightarrow \qquad \mu_n \geq 0$$

Lemma: The "lowest" mode $n = 0$ for a stable static solution belongs to the eigenvalue $\mu_0 = 0$. This "zero"mode is given by $\psi_0(x) = C\,\phi'_{\rm st}(x)$.

Proof: We know $\frac{1}{2}\phi'^{\,2}_{\rm st}(x) = U(\phi_{\rm st}(x))$

$$\partial_x \qquad \Longrightarrow \qquad \phi''_{\rm st}(x) = U'(\phi_{\rm st}(x))$$

$$\partial_x \qquad \Longrightarrow \qquad \phi'''_{\rm st}(x) = U''(\phi_{\rm st}(x))\phi'_{\rm st}(x)$$

Now

$$H\psi_0(x) = C\left[-\partial_x^2 + U''(\phi_{\rm st})\right]\phi'_{\rm st} = C\left(-\phi'''_{\rm st} + U''(\phi_{\rm st})\phi'_{\rm st}\right) = 0$$

Remark: The zero mode is related to the translation invariance

$$\phi_{\rm st}(x + \delta x) = \phi_{\rm st}(x) + \phi'_{\rm st}(x)\delta x = \phi_{\rm st}(x) + \frac{\delta x}{C}\psi_0(x)$$

Fluctuation along zero mode is in essence a translation, here

$$\delta E[\phi_{\rm st}(x + \delta x) - \phi_{\rm st}(x)] = 0 \qquad \text{as} \qquad \mu_0 = 0.$$

5.2 SUSY construction of field models

Recall

$$H = -\partial_x^2 + U''(\phi_{\rm st}(x)) \geq 0$$

with vanishing lowest eigenvalue $\mu_0 = 0$. This allows to interpret

$$H \equiv H_- = -\partial_x^2 + W^2(x) - W'(x)$$

being a Witten partner Hamiltonian with SUSY potential W in units $2m = 1 = \hbar$.
Here choose W such that SUSY is unbroken.

Idea:

- Choose a SUSY potential W, e.g. one of the shape-invariant ones
- Zero mode is given by

$$\psi_0(x) = \mathcal{N}\exp\left\{-\int \mathrm{d}x\, W(x)\right\}$$

- Obtain static solution via integration

$$\phi_{\rm st}(x) = \frac{1}{C}\int \mathrm{d}x\,\psi_0(x)$$

- Use relation

$$U(\phi_{\rm st}(x)) = \frac{1}{2}\phi'^{\,2}_{\rm st}(x)$$

  to obtain an expression $U = U(\phi)$ by eliminating the x via previous relation $\phi_{\rm st} = \phi_{\rm st}(x)$. Choose parameter $\mathcal{N}/C$ most suitable. Finally analytically continue beyond $\phi_\pm$ to $\phi \in \mathbb{R}$.

- A field potential (theory) is found which has a stable static solution. In case of a shape-invariant W we in addition know all the fluctuation modes and their eigenvalues explicitly.

Example: $W(x) = \tanh x$ SUSY partner of free particle, has 1 bound state $\mu_0 = 0$

$$\psi_0(x) = \mathcal{N}\frac{1}{\cosh x} \qquad \text{with} \qquad \mathcal{N}/C = 2$$

$$\phi_{\rm st}(x) = 2\int \mathrm{d}x\,\frac{1}{\cosh x} = 2\arcsin(\tanh x) \qquad \Longrightarrow \qquad \sin\frac{\phi_{\rm st}}{2} = \tanh x$$

$$\phi_{\rm st}(x) \to \phi_\pm = \pm\pi \qquad \text{for} \qquad x \to \pm\infty$$

$$\begin{aligned} U(\phi_{\rm st}) &= \tfrac{1}{2}\phi_{\rm st}'^{\,2}(x) = \tfrac{2}{\cosh^2 x} = 2(1-\tanh^2 x) \\ &= 2(1-\sin^2\tfrac{\phi_{\rm st}}{2}) = 1 + (1-2\sin^2\tfrac{\phi_{\rm st}}{2}) = 1+\cos\phi_{\rm st} \end{aligned}$$

analytical continuation leads to

$$\text{Sine}-\text{Gordon} \qquad U(\phi) = 1+\cos\phi$$

Tutorial: $W(x) = 2\tanh x \quad \Longrightarrow \quad \phi_{\rm st}(x) = \tanh x \quad \Longrightarrow \quad U(\phi) = \frac{1}{2}(1-\phi^2)^2$

Homework: $W(x) = \operatorname{sgn} x \quad \Longrightarrow \quad U(\phi) = \frac{1}{2}(1-|\phi|)^2$

Remarks:

- $W(x) = 3\tanh x \quad \Longrightarrow \quad$ no closed form for U, implicit relations are

$$U(\phi_{\rm st}) = \frac{2}{\cosh^6 x} = U(-\phi_{\rm st})\,, \qquad \phi_{\rm st}(x) = \frac{\tanh x}{\cosh x} + \arcsin(\tanh x)\,, \qquad \phi_\pm = \pm\frac{\pi}{2}$$

- $W(x) = 4\tanh x \quad \Longrightarrow \quad$ new model

$$U(\phi) = \frac{1}{2}\left[1+2\cos\left(\tfrac{2}{3}\arccos(\tfrac{3}{2}\phi)\right)+\tfrac{8\pi}{4}\right]^4\,, \qquad \phi_\pm = \pm\frac{2}{3}$$

- For a complete discussion on shape-inv. SUSY potentials see GJ and P. Roy, Ann. Phys. 256(1997)302. Includes also discussion on unstable fields potentials

6 Supersymmetry in Stochastic Processes

Literature on stochastic processes

1 N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, (North-Holland, 1992)

2 C.W. Gardiner, *Handbook of Stochastic Methods*, (Springer-Verlag, 1990)

6.1 The Langevin Equation

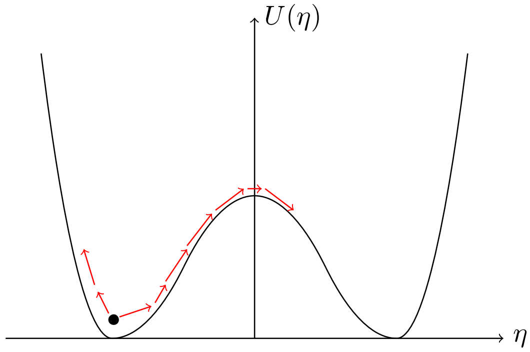
$$\boxed{\dot{\eta} = -U'(\eta) + \xi(t)}$$

Stochastic differential equation where

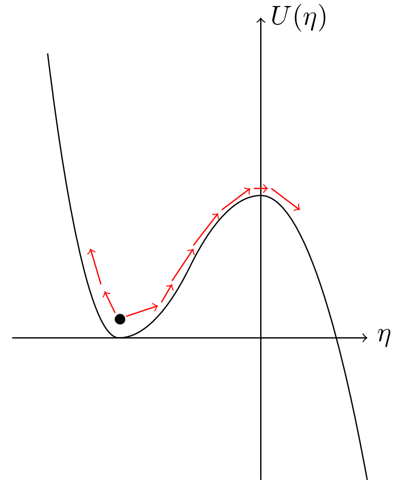
η: macroscopic degree of freedom.
For example, position of a highly overdamped motion of a Brownian particle ($\gamma\dot{\eta} \gg m\ddot{\eta}$)

U: External deterministic force F or drift
$F(\eta) = -U'(\eta)$

ξ: Stochastic force (noise). For example, simulating a coupling to heat bath



(Bi-)stable System



Meta-stable System

Gaussian white noise:

$$\langle \xi(t) \rangle = 0 \qquad \langle \xi(t)\xi(t') \rangle = D\delta(t - t')$$

zero mean — No correlation in time

Diffusion constant D. For ideal heat bath $D = 2k_BT$

Idealisation of more realistic colored noise

$$\langle \xi(t)\xi(t') \rangle = \frac{D}{2\tau_c} \exp\{-|t - t'|/\tau_c\},$$

with correlation time $\tau_c > 0$. Limit $\tau_c \to 0$ = white noise. From now on only white noise.

Average via "path integral":

$$\langle \cdot \rangle := \int_{x(0)=x_0} \mathcal{D}\xi \exp\left\{-\frac{1}{2D}\int_0^\infty d\tau\, \xi^2(\tau)\right\}(\cdot)$$

In general no interest in a particular solution of the Langevin equation, but on average behaviour.

6.2 The Fokker-Planck Equation

Transition probability density:

$$m_t(x, x_0) := \langle \delta(\eta(t) - x) \rangle \qquad \text{where} \qquad x_0 := \eta(0).$$

Is the probability density to arrive at position x at time $t > 0$ for a Brownian particle starting as x_0 at time 0.

Fokker-Planck Equation:

$$\boxed{\frac{\partial}{\partial t} m_t(x, x_0) = \frac{D}{2} \frac{\partial^2}{\partial x^2} m_t(x, x_0) - \frac{\partial}{\partial x} U'(x) m_t(x, x_0)} \qquad (FP)$$

with initial condition $\qquad m_0(x, x_0) = \delta(x - x_0)$.

The stationary distribution:
Assume the below limit exists, then

$$P_{\text{st}}(x) := \lim_{t \to \infty} m_t(x, x_0) \qquad \text{with} \qquad \int_{-\infty}^{+\infty} \mathrm{d}x \, P_{\text{st}}(x) = 1 \, .$$

Insert in (FP):

$$0 = \frac{D}{2} \frac{\partial^2}{\partial x^2} P_{\text{st}}(x) - \frac{\partial}{\partial x} U'(x) P_{\text{st}}(x)$$

Integration:

$$\frac{D}{2} \frac{\partial}{\partial x} P_{\text{st}}(x) - U'(x) P_{\text{st}}(x) = const.$$

As $P_{\text{st}}(x)$ is normalisable we can assume $P_{\text{st}}(x) \to 0$ and $P'_{\text{st}}(x) \to 0$ as $x \to \pm\infty$.
So constant of integration should be $const. = 0$

Integration:

$$\boxed{P_{\text{st}}(x) = C \exp\left\{ -\frac{2}{D} U(x) \right\} = \mathrm{e}^{-U(x)/k_B T}}$$

The assumption that this is normalisable implies restriction on the shape of the drift potential. Typical shapes are

Stable	Meta Stable	Unstable
$P_{\text{st}}(x)$ exists	$\lim_{t \to \infty} m_t(x, x_0) = 0$	$\lim_{t \to \infty} m_t(x, x_0) = 0$