3 The Witten Model

Simple 1-dim. model with $N = 2$ SUSY for quantum particle with mass $m > 0$.

Definitions:

- Hilbert space: $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathbb{C}^2 = \mathcal{H}^+ \oplus \mathcal{H}^-, \quad \mathcal{H}^\pm = L^2(\mathbb{R})$
- Super charge: $Q := \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ with $A := \frac{\mathrm{i}}{\sqrt{2m}} P + \Phi(x) = \frac{\hbar}{\sqrt{2m}} \partial_x + \Phi(x)$
- *SUSY potential*: $\Phi : x \mapsto \Phi(x)$ piecewise continous diff. on $\mathbb{R}$
- Hamiltonian: $H := \{Q, Q^\dagger\} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix}$
- SUSY partner Hamiltonians:

$$\boxed{H_\pm = \frac{P^2}{2m} + \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}} \Phi'(x) = \frac{P^2}{2m} + V_\pm(x)}$$

- SUSY Partner potentials: $V_\pm(x) := \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}} \Phi'(x)$

Remarks:

- Partner Hamiltonians are standard 1-dim. Schrödinger Hamiltonians with special form of the potential
- $\Phi(x) = \sqrt{\frac{m}{2}}\, \omega\, x$ represents the supersymmetric harmonic oscillator
- Conf. space may be replaced by a subspace $\mathcal{M} \subset \mathbb{R}$ with suitable boundary conditions
  $\mathcal{M} = \mathbb{R}^+$ positive half line, radial problems
  $\mathcal{M} = [a, b]$ finite interval, particle in a box

3.1 Ground state for unbroken SUSY

SUSY unbroken $\Longrightarrow$ there exists a $|\phi_0^+\rangle$ and/or $|\phi_0^-\rangle$ such that

$$H_+ |\phi_0^+\rangle = 0 \qquad \text{and/or} \qquad H_- |\phi_0^-\rangle = 0$$

$$\Longleftrightarrow \qquad A^\dagger |\phi_0^+\rangle = 0 \qquad \text{and/or} \qquad A |\phi_0^-\rangle = 0$$

$$\Longleftrightarrow \qquad \left(\mp \tfrac{\hbar}{\sqrt{2m}}\, \partial_x + \Phi(x) \right) \phi_0^\pm(x) = 0$$

$$\Longleftrightarrow \qquad \boxed{\phi_0^\pm(x) = \mathcal{N} \exp\left\{ \pm \frac{\sqrt{2m}}{\hbar} \int_{x_0}^{x} \mathrm{d}z\, \Phi(z) \right\} = \mathcal{N} \mathrm{e}^{\pm U(x)} \in \mathcal{H}^\pm}$$

with *superpotential*:

$$U(x) := \frac{\sqrt{2m}}{\hbar} \int_{x_0}^{x} \mathrm{d}z\, \Phi(z)$$

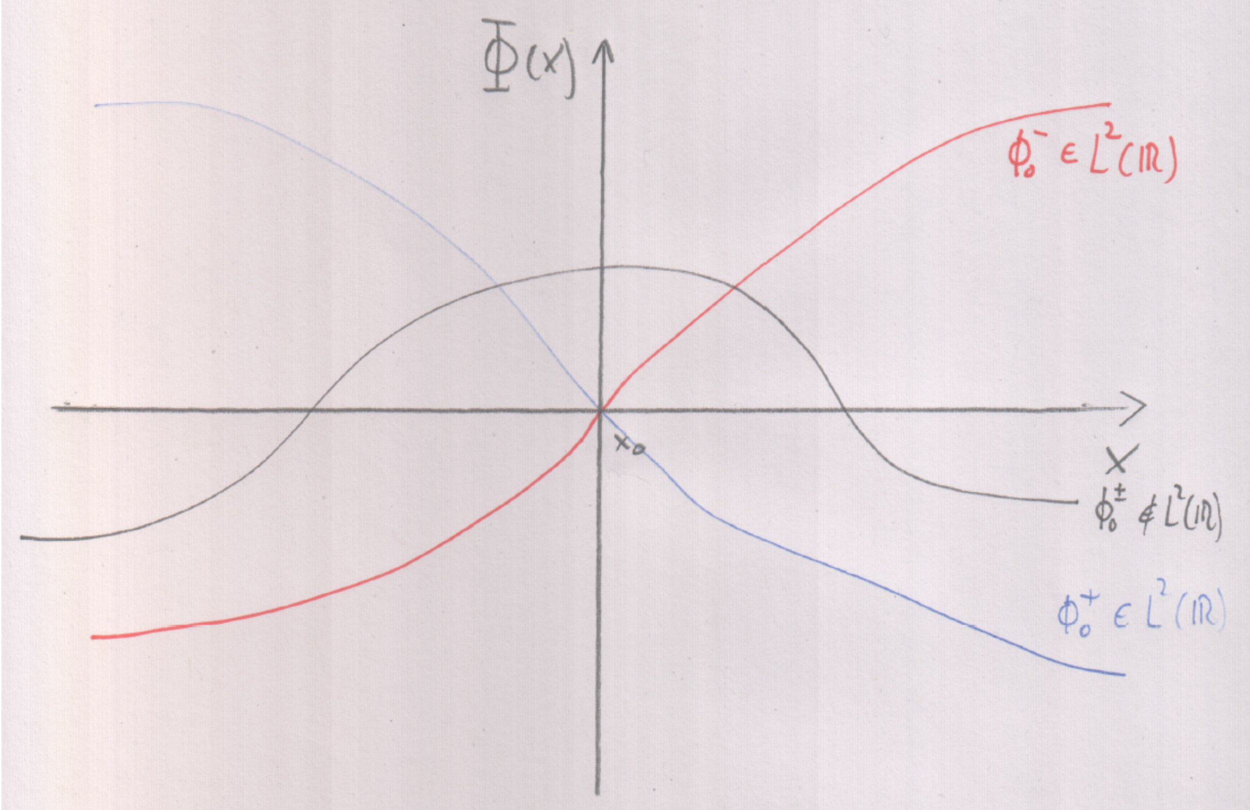
Conclusion:
Only asymptotic behaviour of SUSY potential relevant for SUSY being broken or unbroken!
Let

$$\Phi_\pm := \lim_{x \to \pm\infty} \Phi(x)$$

then

$$\Delta = \frac{1}{2} (\operatorname{sgn} \Phi_+ - \operatorname{sgn} \Phi_-)$$

Witten index is *topological invariant* as it does NOT depend on the details of Φ!



Graphical discussion:

SUSY unbroken: $\Delta = +1$ $\phi_0^- \in \mathcal{H}^-$

SUSY unbroken: $\Delta = -1$ $\phi_0^+ \in \mathcal{H}^+$

SUSY broken: $\Delta = 0$

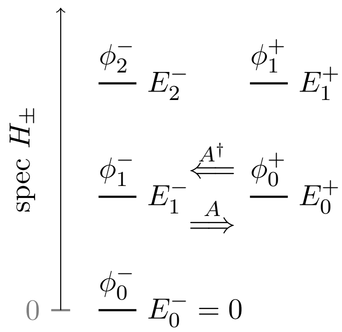
3.2 Spectral properties and SUSY transformations

Let us assume both partner Hamiltonians have a purely discrete spectrum, that is,

$$H_\pm|\phi_n^\pm\rangle = E_n^\pm|\phi_n^\pm\rangle\,, \qquad E_n^\pm < E_{n+1}^\pm\,, \qquad n = 0, 1, 2, 3, \ldots$$

- Good SUSY

$\boxed{\Delta = +1}$



$$E_0^- = 0\,, \qquad E_{n+1}^- = E_n^+ > 0$$

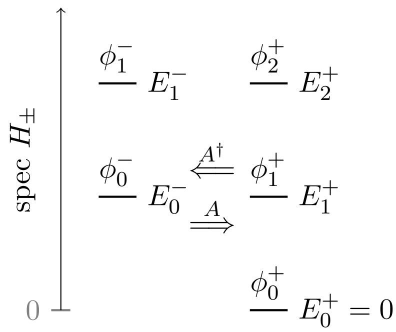
$$\phi_0^-(x) = \mathcal{N}\exp\left\{-\frac{\sqrt{2m}}{\hbar}\int \mathrm{d}x\,\Phi(x)\right\}$$

$$|\phi_{n+1}^-\rangle = \frac{1}{\sqrt{E_n^+}}\,A^\dagger|\phi_n^+\rangle$$

$$|\phi_n^+\rangle = \frac{1}{\sqrt{E_{n+1}^-}}\,A|\phi_{n+1}^-\rangle$$

$$A|\phi_0^-\rangle = 0$$

$\boxed{\Delta = -1}$



$$E_0^+ = 0\,, \qquad E_{n+1}^+ = E_n^- > 0$$

$$\phi_0^+(x) = \mathcal{N}\exp\left\{\frac{\sqrt{2m}}{\hbar}\int \mathrm{d}x\,\Phi(x)\right\}$$

$$|\phi_{n+1}^+\rangle = \frac{1}{\sqrt{E_n^-}}\,A|\phi_n^-\rangle$$

$$|\phi_n^-\rangle = \frac{1}{\sqrt{E_{n+1}^+}}\,A^\dagger|\phi_{n+1}^+\rangle$$

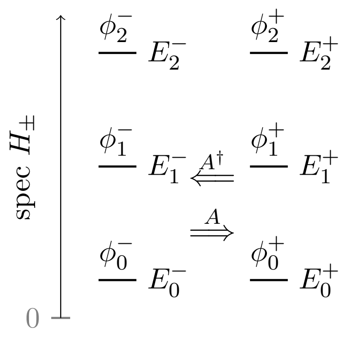
$$A^\dagger|\phi_0^+\rangle = 0$$

Sign convention $\Phi \leftrightarrow -\Phi$ such that for good SUSY $\Delta = +1$

$$\Longrightarrow \qquad \mathrm{sgn}\,\Phi_- < 0 < \mathrm{sgn}\,\Phi_+$$

- Broken SUSY

$\boxed{\Delta = 0}$



$$E_n^- = E_n^+ > 0$$

$$|\phi_n^-\rangle = \frac{1}{\sqrt{E_n^+}}\,A^\dagger|\phi_n^+\rangle$$

$$|\phi_n^+\rangle = \frac{1}{\sqrt{E_n^-}}\,A|\phi_n^-\rangle q$$

Examples:

- Unbroken SUSY $\Delta = +1$

  $$\Phi(x) = \frac{a\hbar}{\sqrt{2m}}\mathrm{sgn}\,(x)|x|^\alpha\,, \qquad a > 0\,, \qquad \alpha > 0$$

  $$V_\pm(x) = \frac{\hbar^2 a^2}{2m}\left(|x|^{2\alpha} \pm \frac{\alpha}{a}|x|^{\alpha-1}\right)$$ [draw graphs]

  $$\phi_0^-(x) = \mathcal{N}\exp\left\{-\tfrac{a}{\alpha+1}|x|^{\alpha+1}\right\}$$

- Broken SUSY $\Delta = 0$
  $\Phi(x) = \frac{a\hbar}{\sqrt{2m}}|x|^\alpha\,, \qquad a > 0\,, \qquad \alpha > 0$
  $V_\pm(x) = \frac{\hbar^2 a^2}{2m}\left(|x|^{2\alpha} \pm \frac{\alpha}{a}\mathrm{sgn}\,(x)|x|^{\alpha-1}\right)$ [draw graphs]
- More examples $\Longrightarrow$ Tutorial

3.3 Shape invariance and exact solutions

Assumption: SUSY potential depends on some parameter a, that is,

$$\Phi(\cdot, x) : a \mapsto \Phi(a, x)\,, \qquad a \in I \subseteq \mathbb{R}$$

Hence,

$$V_\pm(a, x) = \Phi^2(a, x) \pm \frac{\hbar}{\sqrt{2m}}\frac{\partial}{\partial x}\Phi(a, x).$$

Definition: The partner potentials $V_\pm(a_0, x)$ are called *shape-invariant* if they are related by

$$\boxed{V_+(a_0, x) = V_-(a_1, x) + R(a_1)\,, \qquad \forall\, x \in \mathbb{R}\,,}$$

where a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping $F : a_0 \mapsto a_1 = F(a_0)$ and the residual term $R(a_1)$ is independent of the variable x.

Example:

$$\Phi(a, x) := \frac{\hbar}{\sqrt{2m}}\,a\tanh x, \qquad a > 0.$$

$$V_\pm(a, x) = \frac{\hbar^2}{2m}\left[a^2 - \frac{a(a \mp 1)}{\cosh^2 x}\right]$$

Obviously

$$V_+(a_0, x) = V_-(a_0 - 1, x) + \frac{\hbar^2}{2m}\left[a_0^2 - (a_0 - 1)^2\right].$$

Therefore

$$a_1 = F(a_0) = a_0 - 1, \qquad R(a_1) = \frac{\hbar^2}{2m}\left[a_0^2 - a_1^2\right] = \frac{\hbar^2}{2m}\left[a_0^2 - (a_0 - 1)^2\right] > 0\,,$$

and

$$\phi_0^-(a_0, x) = \frac{\mathcal{N}}{\cosh^{a_0} x}.$$

Let us assume we have a family of pairwise shape invariant potentials

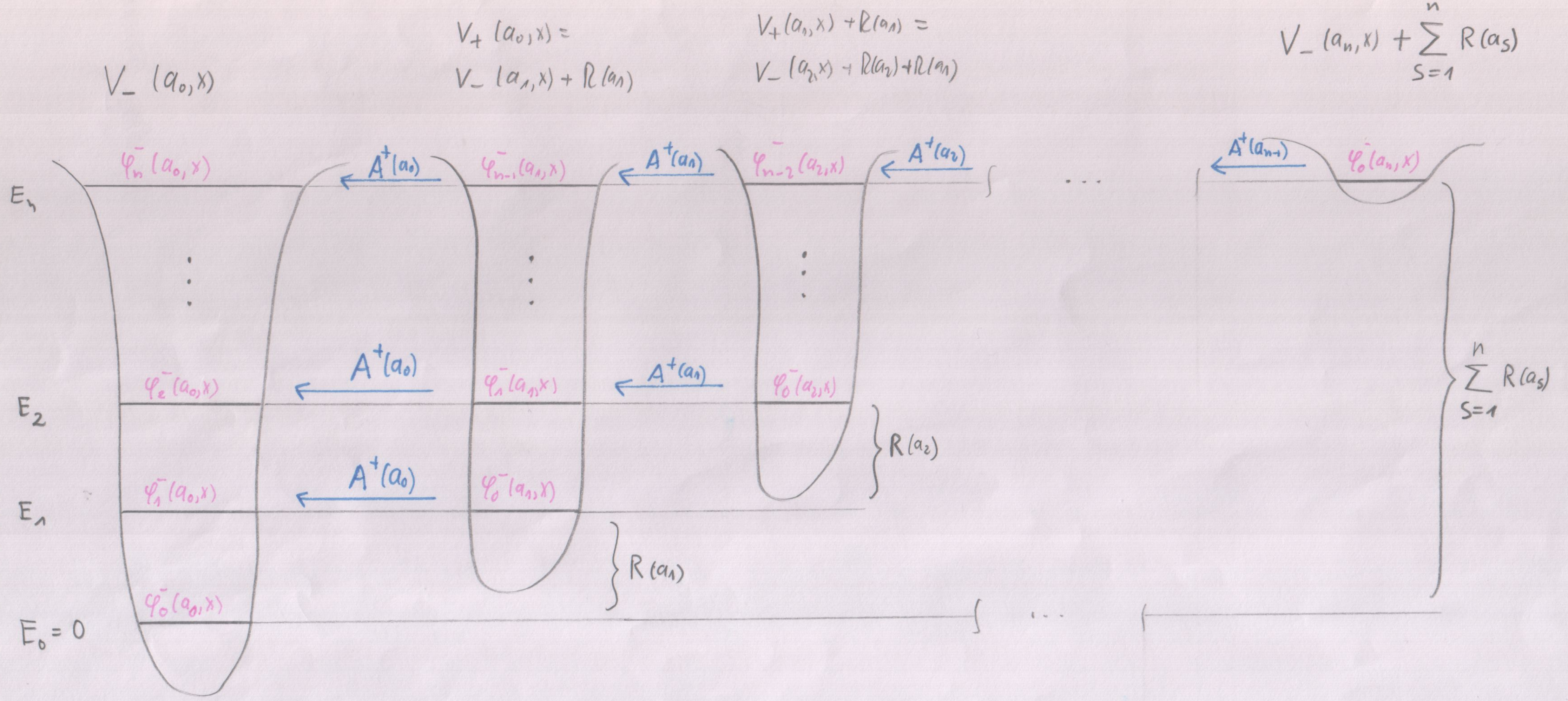
$$\{\Phi(a_s, x)\}\,, \qquad s = 0, 1, 2, \ldots, n$$

such that for all $\Delta = +1$.
Some obvious relations follow from graph on next page:

$$E_0 = 0\,, \qquad E_n = \sum_{s=1}^{n} R(a_s)$$

$$\phi_0^-(a_s, x) = \mathcal{N}\exp\left\{-\frac{\sqrt{2m}}{\hbar}\int_0^x \mathrm{d}z\,\Phi(a_s, z)\right\}$$

$$\Phi_{n-s}^-(a_s, x) = \frac{1}{\sqrt{E_n - E_s}}\,A^\dagger(a_s)\Phi_{n-(s+1)}^-(a_s, x)$$



$$A^+(a_s) := -\frac{\hbar}{\sqrt{2m}}\,\partial_x + \Phi(a_s,x)$$

Conclusion: Spectral properties of $H = \frac{P^2}{2m} + V_-(a_0, x)$ are given by

$$E_n = \sum_{s=1}^{n} R(a_s),$$
$$\phi_n^-(a_0, x) = \frac{A^\dagger(a_0)}{[E_n - E_0]^{1/2}} \cdots \frac{A^\dagger(a_{n-1})}{[E_n - E_{n-1}]^{1/2}} \, \phi_0^-(a_n, x),$$
$$\phi_0^-(a_n, x) = \mathcal{N} \exp\left\{ -\frac{\sqrt{2m}}{\hbar} \int_0^x \mathrm{d}z \, \Phi(a_n, z) \right\},$$

with

$$A^\dagger(a_s) := -\frac{\hbar}{\sqrt{2m}} \frac{\partial}{\partial x} + \Phi \,, \qquad \text{and} \qquad a_s = F(a_{s-1}(a_s, x)$$

The eigenvalue problem (discrete part) of a family of shape invariant Hamiltonians is exactly solvable

Remark:
This is not a new result due to SUSY QM but basically the well-known Schrödinger-Infeld-Hull factorization method [Rev. Mod. Phys. 23 (1951) 21]

Table of shape invariant potentials:

SUSY potential $\Phi(x)/\frac{\hbar}{\sqrt{2m}}$	config. space[a]	parameter range for good SUSY[b]	partner potentials $V_\pm(x)/\frac{\hbar^2}{2m}$
$A\tanh x + B/\cosh x$	$\mathbb{R}$	$A > 0$	$A^2 + \frac{B^2 - A(A\mp1) + B(2A\mp1)\sinh x}{\cosh^2 x}$
$A\coth x - B/\sinh x$	$\mathbb{R}^+$	$B > A > 0$	$A^2 + \frac{B^2 + A(A\mp1) - B(2A\pm1)\cosh x}{\sinh^2 x}$
$-A\cot x + B/\sin x$	$[0, \pi]$	$A > B > 0$	$-A^2 + \frac{B^2 + A(A\pm1) - B(2A\mp1)\cos x}{\sin^2 x}$
$A\tan x - B\cot x$	$[0, \pi/2]$	$A > 0,\ B > 0$[c]	$-(A+B)^2 + \frac{A(A\pm1)}{\cos^2 x} + \frac{B(B\pm1)}{\sin^2 x}$
$A\tanh x - B\coth x$	$\mathbb{R}^+$	$A > B > 0$[c]	$(A-B)^2 - \frac{A(A\mp1)}{\cosh^2 x} + \frac{B(B\pm1)}{\sinh^2 x}$
$A\tanh x + B/A$	$\mathbb{R}$	$A > B \geq 0$	$A^2 + \frac{B^2}{A^2} - \frac{A(A\mp1)}{\cosh^2 x} + 2B\tanh x$
$-A\coth x + B/A$	$\mathbb{R}^+$	$B > A > 0$	$A^2 + \frac{B^2}{A^2} + \frac{A(A\pm1)}{\sinh^2 x} - 2B\coth x$
$-A\cot x + B/A$	$[0, \pi]$	$A > 0$	$-A^2 + \frac{B^2}{A^2} + \frac{A(A\pm1)}{\sin^2 x} - 2B\cot x$
$Ax - B/x$	$\mathbb{R}^+$	$A > 0,\ B > 0$[c]	$-A(2B \mp 1) + A^2x^2 + \frac{B(B\pm1)}{x^2}$
$-A/x + B/A$	$\mathbb{R}^+$	$A > 0,\ B > 0$	$\frac{B^2}{A^2} - \frac{2B}{x} + \frac{A(A\pm1)}{x^2}$
$-A\mathrm{e}^{-x} + B$	$\mathbb{R}$	$A > 0,\ B > 0$	$B^2 + A^2\mathrm{e}^{-2x} - A(2B \mp 1)\mathrm{e}^{-x}$
$Ax + B$	$\mathbb{R}$	$A > 0$	$(Ax+B)^2 \pm A$

[a] For $x \in \mathbb{R}^+$, $x \in [0, \pi/2]$, and $x \in [0, \pi]$ we impose Dirichlet boundary conditions on the wave functions at $x = 0$, $x = 0, \pi/2$, and $x = 0, \pi$, respectively.
[b] With our convention that the ground state is an eigenstate of H_-.
[c] These examples belong to class 2 of Gendenstheîn and will give rise to a broken SUSY potential if B is replaced by $-B$.

Our Example:

$$\begin{aligned}
\Phi(a,x) &:= \frac{\hbar}{\sqrt{2m}}\, a \tanh x, \qquad a>0\\
V_{\pm}(a_0,x) &= \tfrac{\hbar^2}{2m}\left[a_0^2 - \tfrac{a_0(a_0\pm 1)}{\cosh^2 x}\right]\\
a_s &= F(a_{s-1}) = a_{s-1}-1 = a_0 - s\\
R(a_s) &= \frac{\hbar^2}{2m}\left[a_{s-1}^2 - a_s^2\right],
\end{aligned}$$

SUSY ground state normalizable for $n < a_0$

$$\phi_0^-(a_n,x) = \phi_0^-(a_0-n,x) = C\cosh^{n-a_0} x$$

Eigenvalues

$$E_n = \frac{\hbar^2}{2m}\sum_{s=1}^{n}(a_{s-1}^2 - a_s^2) = \frac{\hbar^2}{2m}[a_0^2-(a_0-n)^2], \qquad n = 0,1,2,\ldots < a_0.$$

Eigenfunctions

$$\phi_n^-(a_0,x) = C_n\left[-\partial_x + a_0\tanh x\right]\cdots\left[-\partial_x + (a_0-n+1)\tanh x\right]\cosh^{n-a_0} x$$

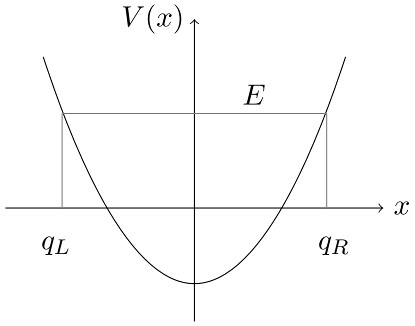
where

$$C_n := \mathcal{N}\prod_{s=0}^{n-1}\left[(a_0-s)^2-(a_0-n)^2\right]^{-1/2}, \qquad n = 1,2,3,\ldots < a_0.$$

3.4 Quasi classical approximation

3.4.1 The WKB approximation

Consider single-well potential with classical left and right turning points $q_L(E)$ and $q_R(E)$ for given energy E: $\qquad E = V(q_L) = V(q_R)$



The WKB formula (good for small $\hbar$) reads

$$\int_{q_L}^{q_R} \mathrm{d}x\,\sqrt{2m(E-V(x))} = \hbar\pi\left(n+\tfrac{1}{2}\right), \qquad n = 0,1,2,3,\ldots$$

and provides an approximation for the quantum energy eigenvalues E_n

Remarks:

- In general a good approximation for large n.
- For the ground state energy less useful.

3.4.2 The SUSY version

Consider

$$V_{\pm}(x) = \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}}\,\Phi'(x)$$

and interpret last term as quantum correction to the classical potential $V_{\text{class}}(x) = \Phi^2(x)$. Apply WKB formula

$$\begin{aligned}
I &:= \sqrt{2m}\int_{q_L}^{q_R} \mathrm{d}x\,\sqrt{E - \Phi^2(x) \mp \frac{\hbar}{\sqrt{2m}}\,\Phi'(x)} \\
&\approx \sqrt{2m}\int_{x_L}^{x_R} \mathrm{d}x\,\sqrt{E-\Phi^2(x)}\left(1 \mp \frac{\hbar}{\sqrt{2m}}\,\frac{1}{2}\,\frac{\Phi'(x)}{\sqrt{E-\Phi^2(x)}}\right) \\
&\qquad \text{here} \qquad q_{L/R} \to x_{L/R} \qquad \text{where} \qquad \Phi^2(x_{L/R}) = E \\
&= \int_{x_L}^{x_R} \mathrm{d}x\,\sqrt{2m(E-\Phi^2(x))} \mp \frac{\hbar}{2}\int_{x_L}^{x_R}\mathrm{d}x\,\frac{\Phi'(x)}{\sqrt{E-\Phi^2(x)}} \\
&= \int_{x_L}^{x_R} \mathrm{d}x\,\sqrt{2m(E-\Phi^2(x))} \mp \frac{\hbar}{2}\underbrace{\int_{\Phi(x_L)}^{\Phi(x_R)}\mathrm{d}\Phi\,\frac{1}{\sqrt{E-\Phi^2}}}_{=:J}
\end{aligned}$$

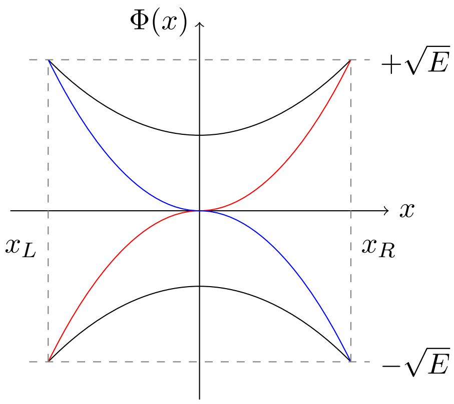
Four cases for the integral J

$$J = \int_{\Phi(x_L)}^{\Phi(x_R)} \mathrm{d}\Phi\,\frac{1}{\sqrt{E-\Phi^2}} = \arcsin\frac{\Phi(x_R)}{\sqrt{E}} - \arcsin\frac{\Phi(x_L)}{\sqrt{E}}$$

Case $\Delta = 0$**:** $\Phi(x_L) = \Phi(x_R) = \pm\sqrt{E} \quad\Longrightarrow\quad J = 0$

Case $\Delta = +1$**:** $\Phi(x_L) = -\Phi(x_R) = -\sqrt{E} \quad\Longrightarrow\quad J = +\pi$

Case $\Delta = -1$**:** $\Phi(x_L) = -\Phi(x_R) = +\sqrt{E} \quad\Longrightarrow\quad J = -\pi$



Result:

$$I = \int_{x_L}^{x_R}\mathrm{d}x\,\sqrt{2m(E-\Phi^2(x))} \mp \frac{\hbar}{2}\,\pi\Delta$$

Hence, via WKB formula we arrive at the *Supersymmetric version of WKB* for both $V_{\pm}(x)$

$$\boxed{\int_{x_L}^{x_R}\mathrm{d}x\,\sqrt{2m(E-\Phi^2(x))} = \hbar\pi\left(n + \frac{1}{2} \pm \frac{\Delta}{2}\right)}$$

Remarks:

- $\Delta = +1$: $E_0^- = 0$ is exact! $E_{n+1}^- = E_n^+ > 0$ spectral symmetry conserved!
- $\Delta = -1$: $E_0^+ = 0$ is exact! $E_{n+1}^+ = E_n^- > 0$ spectral symmetry conserved!
- $\Delta = 0$: $E_n^+ = E_n^- > 0$ spectral symmetry conserved!
- For all shape invariant potentials ALL $E_n^\pm$ are exact for all Δ!
- For other systems supersymmetric version usually provides better approximations!

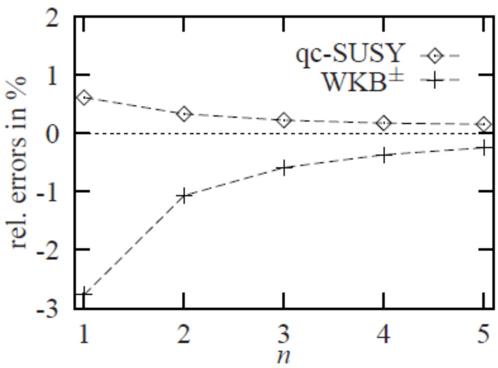


Figure 6.12: Relative errors for the good-SUSY potential $\Phi(x) = \sqrt{\hbar^2/2m}\,\sinh x$. Here the WKB approximation respects the exact relation $E_n^- = E_{n+1}^+$.

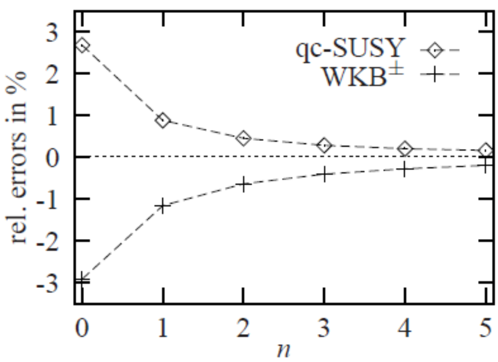


Figure 6.13: Relative errors for the broken-SUSY potential $\Phi(x) = \sqrt{\hbar^2/2m}\,\cosh x$.

Summary of section 3

1-dim. Witten model fully characterised by SUSY potential Φ
Partner potentials

$$V_{\pm}(x) = \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}}\,\Phi'(x)$$

Witten index given by

$$\Delta = \frac{1}{2}\,(\Phi_+ - \Phi_-)$$

Shape invariance

$$V_+(a_0, x) = V_-(a_1, x) + R(a_1)$$

provides exact solutions
Quasi classical SUSY approximation for spectrum of $H_{\pm}$

$$\int_{x_L}^{x_R} \mathrm{d}x\,\sqrt{2m(E - \Phi^2(x))} = \hbar\pi\left(n + \frac{1}{2} \pm \frac{\Delta}{2}\right)$$

SUSY transformations for continuous states (scattering) $\Longrightarrow$ Tutorial Exercise 5