Supersymmetric Quantum Mechanics

Lecture Notes

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Preliminaries

Dates:
Six Mondays 17.04.23, 24.04.23, 08.05.23, 15.05.23, 22.05.23, 29.05.23, 05.06.23 (Test?)
Lecture 9 - 12, Tutorial 13 - 15, Homework Problems
Script and other details are available at
`https://www.eso.org/~gjunker/VorlesungSS2023.html`

Literature:

- Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer-Verlag) 1st edition
- Kalka H and Soff G 1997 *Supersymmetrie* (Stuttgart: Teubner)
- Cooper F, Khare A and Sukhatme U 2001 *Supersymmetry in Quantum Mechanics* (Singapore: World Scientific)
- Bagchi B 2001 *Supersymmetry in Quantum and Classical Mechanics* (Boca Raton: Chapman & Hall/CRC)
- Gangopadhyaya A, Mallow J V and Rasinariu C 2011 *Supersymmetric Quantum Mechanics: An Introduction* (Singapore: World Scientific)
- Junker G 2019, *Supersymmetric Methods in Quantum, Statistical and Solid State Physics* (Bristol: IOP) $\Longrightarrow$ "The Book"
- ...

Supersymmetric Quantum Mechanics:

$$\text{SUSY QM} = \text{QM} + \text{Supercharges}$$

Supercharges are conserved quantities obeying a SUSY algebra

Aim of lecture:
Supersymmetry (SUSY) as an algebraic tool with many applications in theoretical and mathematical physics and beyond.

1 Historical Background

SUSY idea originates in quantum field theory (gauge theories)

• Structure:

Space – Time Sym. (Poincare Algebra)	Internal (Gauge) Sym. (Lie Algebra)
Matter Fields (Fermions)	Gauge Fields (Bosons)

• SUSY idea: Unify space-time and internal symmetries

$\Longrightarrow$ Unification of Fermions and Bosons

NoGo-Theorem of Coleman and Mandula
Within the context of Lie algebras NOT possible

$\Longrightarrow$ Super (or graded) Lie algebras close under

$$\text{Commutator} \qquad [A,B] := AB - BA$$

and

$$\text{Anticommutator} \qquad \{A,B\} := AB + BA$$

• 1976: H. Nicolai invented SUSY QM as $(0+1)$-dim. QFT

• 1981: E. Witten introduced a simple QM model (Witten model)
$\Longrightarrow$ popularization

• More background is given in "The Book"

Content

- Supersymmetric Quantum Mechanics (definitions and properties)
- The Witten Model (non-relativistic SUSY QM)
- Darboux Method (construct problem with known solution)
- Classical Field in $(1+1)$ Dimensions (SUSY in classical systems)
- Supersymmetry in Stochastic Processes (SUSY in classical stochastic systems)
- Supersymmetry and Pauli Hamiltonians (Aharonov-Casher; Pauli paramagnetism)
- Supersymmetry and Dirac Hamiltonians (SUSY in rel. QM systems; Graphene)

2 Supersymmetric Quantum Mechanics

2.1 Definitions

Assumptions:

Hilbert space:	$\mathcal{H}$	
Hamiltonian:	$H = H^\dagger$	
Observables:	$Q_i = Q_i^\dagger,$	$i = 1, 2, 3 \ldots, N$

Definition 2.1: A quantum system characterised by the set $\{H, Q_1, \ldots, Q_N; \mathcal{H}\}$, is called *supersymmetric* if the following anticommutation relation is valid for all $i, j = 1, 2, \ldots, N$:

$$\boxed{\{Q_i, Q_j\} = H\delta_{ij},} \tag{1}$$

where δ_{ij} denotes Kronecker's delta symbol. The self-adjoint operators Q_i are called *supercharges* and the Hamiltonian H is called *SUSY Hamiltonian.* The symmetry described by the *superalgebra* (1) is called *N-extended supersymmetry.*

Remarks:

- $H = 2Q_1^2 = 2Q_2^2 = \cdots = 2Q_N^2 = \dfrac{2}{N}\displaystyle\sum_{i=1}^{N} Q_i^2 \geq 0$
  no negative energy eigenvalues
  $Q_i = \sqrt{H/2}$ square root of Hamiltonian

- $[H, Q_i] = 0$ *supercharges* Q_i are constants of motion if $\dfrac{\partial Q_i}{\partial t} = 0$

- For $N \geq 2$ we may introduce complex supercharges

$$\tilde{Q}_k := \frac{1}{\sqrt{2}}\left[Q_{2k-1} + \mathrm{i}Q_{2k}\right]$$

$$\boxed{\left\{\widetilde{Q}_k, \widetilde{Q}_l^\dagger\right\} = H\delta_{kl}\,, \qquad \widetilde{Q}_k^2 = 0 = \left(\widetilde{Q}_k^\dagger\right)^2}$$

  Show that $\left\{\widetilde{Q}_k, \widetilde{Q}_l\right\} = 0$ for all k, l.

Let $E_0 := \inf \operatorname{spec} H \geq 0$ be ground state energy of H with

$$H|\psi_0^j\rangle = E_0|\psi_0^j\rangle\,, \qquad j = 1, 2, 3, \ldots, g \qquad (g = \text{degeneracy of } E_0)$$

Definition:

$$\boxed{\begin{array}{ll} \text{SUSY unbroken :} & \Longleftrightarrow E_0 = 0 \\ \text{SUSY broken :} & \Longleftrightarrow E_0 > 0 \end{array}}$$

Remarks: $E_0 = \langle\psi_0^j|H|\psi_0^j\rangle = \dfrac{2}{N}\displaystyle\sum_{i=1}^{N}\langle\psi_0^j|Q_i^2|\psi_0^j\rangle = \dfrac{2}{N}\displaystyle\sum_{i=1}^{N}||Q_i|\psi_0^j\rangle||^2$

$E_0 = 0 \qquad \Longleftrightarrow \qquad Q_i|\psi_0^j\rangle = 0 \qquad$ for all (i, j)

$E_0 > 0 \qquad \Longleftrightarrow \qquad \exists$ pair (i, j) such that $Q_i|\psi_0^j\rangle \neq 0$

ground state is NOT invariant under SUSY transformations and SUSY is broken

2.2 The Supersymmetric Harmonic Oscillator

Consider 1-dim. quantum particle with spin $\frac{1}{2}$ and unit mass $m = 1$

Hilbert space: $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$

"Bosonic" degree of freedom: $a := \frac{1}{\sqrt{2}}(\partial_x + x) \quad \Longrightarrow \quad [a, a^\dagger] = 1$

"Fermionic" degree of freedom: $b := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \Longrightarrow \quad \{b, b^\dagger\} = 1\,, \quad b^2 = 0 = (b^\dagger)^2$

Complex Supercharge: Use no longer tilde $\tilde{Q} \equiv Q$

$$Q := a \otimes b^\dagger = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\,, \qquad Q^\dagger = a^\dagger \otimes b = \begin{pmatrix} 0 & 0 \\ a^\dagger & 0 \end{pmatrix}$$

SUSY Hamiltonian:

$$\begin{aligned} H \ &:= \{Q, Q^\dagger\} = a^\dagger a + b^\dagger b \\ &= \tfrac{1}{2}(-\partial_x^2 + x^2 - 1) \otimes 1 + 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \tfrac{1}{2}(-\partial_x^2 + x^2) \otimes 1 + 1 \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \end{aligned}$$

Spectral properties of H:

- Eigenstates

$$|n, \downarrow\rangle := |n\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,, \qquad |n, \uparrow\rangle := |n\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \qquad n \in \mathbb{N}_0\,,$$

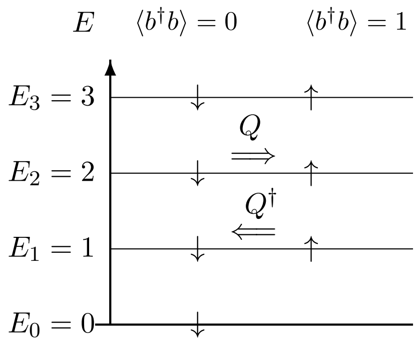
  where

$$a|n\rangle = \sqrt{n}|n-1\rangle\,, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

  and

$$b|n,\uparrow\rangle = |n,\downarrow\rangle\,, \qquad b|n,\downarrow\rangle = 0\,, \qquad b^\dagger|n,\downarrow\rangle = |n,\uparrow\rangle\,, \qquad b^\dagger|n,\uparrow\rangle = 0\,.$$

- Eigenvalues: $\langle a^\dagger a\rangle = n\,, \quad n = 0, 1, 2, 3, \ldots$



- SUSY unbroken as $E_0 = 0$
- $E > 0$ pairwise degenerate

SUSY Transformations:

$$Q|n,\downarrow\rangle = \sqrt{n}|n-1,\uparrow\rangle\,, \qquad Q|n,\uparrow\rangle = 0\,,$$

$$Q^\dagger|n,\uparrow\rangle = \sqrt{n+1}|n+1,\downarrow\rangle\,, \qquad Q^\dagger|n,\downarrow\rangle = 0\,.$$

Q and $Q^\dagger$ transform between spin-down and spin-up state with SAME energy eigenvalue. Is generic property for all $N \geq 2$ SUSY QM systems as there exists a Witten parity operator.

2.3 Properties of $N = 2$ SUSY QM

Consider $N = 2$ SUSY QM: $\{H, Q_1, Q_2; \mathcal{H}\}$

Recall: $Q_1 Q_2 = -Q_2 Q_1, \quad H = 2Q_1^2 = 2Q_2^2 = Q_1^2 + Q_2^2$

Complex Supercharge: $Q := \frac{1}{\sqrt{2}}(Q_1 + \mathrm{i}Q_2), \quad Q^\dagger = \frac{1}{\sqrt{2}}(Q_1 - \mathrm{i}Q_2)$

SUSY algebra:

$$\boxed{Q^2 = 0 = (Q^\dagger)^2 \,, \qquad \{Q, Q^\dagger\} = H}$$

2.3.1 The Witten parity

Let us assume there exists a self-adjoint operator W such that

$$\boxed{[W, H] = 0, \qquad \{W, Q\} = 0 = \{W, Q^\dagger\}, \qquad W^2 = \mathbf{1}.}$$

Definition: A self-adjoint operator W which obeys above relations is called *Witten parity* or *Witten operator*. The quantum system $\{H, Q, Q^\dagger, W; \mathcal{H}\}$ will be called a *supersymmetric quantum system with Witten parity.*

Remarks: See Tutorial Exercise 2 and 3

- $\operatorname{spec} W = \{-1, +1\}$ non-trivial unitary involution on $\mathcal{H}$
  $[Q, H] = 0 = [Q^\dagger, H]$ constant of motion
- For $N \geq 2$ formal construction on $\mathcal{H} \backslash \ker(H)$ via

$$\boxed{W := \frac{2}{H} Q Q^\dagger - \mathbf{1} = \frac{1}{\mathrm{i}H}\,[Q_1, Q_2] = \frac{[Q, Q^\dagger]}{\{Q, Q^\dagger\}}}$$

- "Fermionic" annihilation operator

$$b := Q^\dagger / \sqrt{H} \qquad \text{on} \qquad \mathcal{H} \backslash \ker(H)$$

  obeying the relations

$$\{b, b^\dagger\} = \mathbf{1}, \qquad b^2 = 0 = \left(b^\dagger\right)^2 .$$

- "Fermion" number operator

$$\mathcal{F} := b^\dagger b = Q Q^\dagger / H = \mathcal{F}^\dagger = \mathcal{F}^2$$

  obeys the algebra

$$[\mathcal{F}, H] = 0 \,, \qquad [\mathcal{F}, Q] = Q \,, \qquad [\mathcal{F}, Q^\dagger] = -Q^\dagger \,,$$

  and is related to the Witten parity by

$$W = 2\mathcal{F} - \mathbf{1} = (-\mathbf{1})^{\mathcal{F}+1} .$$

2.3.2 Witten parity subspaces

Definition: Let $P^\pm := \frac{1}{2}(1 \pm W)$ be the orthogonal projection of $\mathcal{H}$ onto the eigenspace of the Witten operator with eigenvalue ± 1, respectively.
The subspace

$$\mathcal{H}^\pm := P^\pm \mathcal{H} P^\pm = \{|\psi\rangle \in \mathcal{H} \,:\, W|\psi\rangle = \pm|\psi\rangle\}$$

is called space of *positive* ($\mathcal{H}^+$) and *negative* ($\mathcal{H}^-$) Witten parity, respectively.

Remarks:

- Projectors:

$$P^{\pm}P^{\pm} = \tfrac{1}{4}(1 \pm W)(1 \pm W) = \tfrac{1}{4}(1 \pm 2W + W^2) = \tfrac{1}{2}(1 \pm W) = P^{\pm} \qquad \text{projector}$$

$$P^{\pm}P^{\mp} = \tfrac{1}{4}(1 \pm W)(1 \mp W) = \tfrac{1}{4}(1 - W^2) = 0 \qquad \text{orthogonal}$$

$$P^{+} + P^{-} = 1 \qquad \text{complete}$$

$\Longrightarrow \qquad \mathcal{H} = \mathcal{H}^{+} \oplus \mathcal{H}^{-} \qquad$ grading of $\mathcal{H}$ induced by W.

- Matrix representation

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad P^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P^{-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$|\psi^{+}\rangle = \begin{pmatrix} |\phi^{+}\rangle \\ 0 \end{pmatrix}, \qquad |\psi^{-}\rangle = \begin{pmatrix} 0 \\ |\phi^{-}\rangle \end{pmatrix}, \qquad |\phi^{\pm}\rangle \in \mathcal{H}^{\pm}$$

- Supercharges: $\{Q, W\} = 0$

$\Longrightarrow \qquad \pm Q|\psi^{\pm}\rangle = QW|\psi^{\pm}\rangle = -WQ|\psi^{\pm}\rangle \qquad \Longrightarrow \qquad WQ|\psi^{\pm}\rangle = \mp Q|\psi^{\pm}\rangle$

Hence $\qquad Q|\psi^{\pm}\rangle \in \mathcal{H}^{\mp} \qquad$ or $\qquad Q|\psi^{\pm}\rangle = 0$

Similar $\qquad Q^{\dagger}|\psi^{\pm}\rangle \in \mathcal{H}^{\mp} \qquad$ or $\qquad Q^{\dagger}|\psi^{\pm}\rangle = 0$

Q and $Q^{\dagger}$ transform between $\mathcal{H}^{+}$ and $\mathcal{H}^{-} \qquad \Longrightarrow \qquad$ *SUSY transformations*

Hence $Q\mathcal{H}^{-} \subset \mathcal{H}^{+}, \qquad Q^{\dagger}\mathcal{H}^{+} \subset \mathcal{H}^{-}$

- Without loss of generality (see Tutorial Exercise 4):

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q^{\dagger} = \begin{pmatrix} 0 & 0 \\ A^{\dagger} & 0 \end{pmatrix},$$

with $\qquad A : \mathcal{H}^{-} \to \mathcal{H}^{+} \qquad$ and $\qquad A^{\dagger} : \mathcal{H}^{+} \to \mathcal{H}^{-}$

Observe $\qquad Q^{\dagger}\mathcal{H}^{-} = 0 = Q\mathcal{H}^{+}$

- SUSY partner Hamiltonians:

$$H = Q^{\dagger}Q + QQ^{\dagger} = \begin{pmatrix} AA^{\dagger} & 0 \\ 0 & A^{\dagger}A \end{pmatrix} = \begin{pmatrix} H_{+} & 0 \\ 0 & H_{-} \end{pmatrix}$$

with *SUSY partner Hamiltonians*

$$H_{+} := AA^{\dagger} \geq 0, \qquad H_{-} := A^{\dagger}A \geq 0.$$

- Even and odd operators:
An arbitrary operator O acting on $\mathcal{H}$ can be decomposed into its diagonal (even) part O_{e} and its off-diagonal (odd) part O_{o}. That is, $O = O_{\mathrm{e}} + O_{\mathrm{o}}$ with

$$[W, O_{\mathrm{e}}] = 0, \qquad \{W, O_{\mathrm{o}}\} = 0.$$

In general

$$O = \begin{pmatrix} O_{++} & O_{+-} \\ O_{-+} & O_{--} \end{pmatrix}$$

with O_{++} and O_{--} forming the even part and O_{+-} and O_{-+} the odd part of O.
In particular, the SUSY Hamiltonian H is an even operator, whereas the supercharges Q and $Q^{\dagger}$ are odd operators.

2.3.3 SUSY Transformations

Definition: Eigenstates of W are called *positive* and *negative (Witten-) parity states*, respectively. They are denoted by $|\psi^\pm\rangle$:

$$W|\psi^\pm\rangle = \pm|\psi^\pm\rangle\,, \qquad |\psi^\pm\rangle \in \mathcal{H}^\pm\,.$$

For simplicity we will call them also positive and negative states.

Proposition: To each positive (negative) eigenstate $|\psi_E^+\rangle$ ($|\psi_E^-\rangle$) of the SUSY Hamiltonian H with eigenvalue $E > 0$ there exists a negative (positive) eigenstate of H with the same eigenvalue. These eigenstates are related by the *SUSY transformations*

$$\boxed{|\psi_E^-\rangle = \frac{1}{\sqrt{E}}\,Q^\dagger|\psi_E^+\rangle, \qquad |\psi_E^+\rangle = \frac{1}{\sqrt{E}}\,Q|\psi_E^-\rangle,}$$

where

$$W|\psi_E^\pm\rangle = \pm|\psi_E^\pm\rangle \qquad \text{and} \qquad H|\psi_E^\pm\rangle = E|\psi_E^\pm\rangle\,.$$

Proof: As $[W,H]=0 \qquad \Longrightarrow \qquad$ common eigenbasis

Let $H|\psi_E^-\rangle = E|\psi_E^-\rangle \qquad \Longrightarrow \qquad HQ|\psi_E^-\rangle = QH|\psi_E^-\rangle = EQ|\psi_E^-\rangle \in \mathcal{H}^+$.

$\Longrightarrow \qquad |\psi_E^+\rangle := \frac{1}{\sqrt{E}}Q|\psi_E^-\rangle$ is positive eigenstate of H for the same eigenvalue $E > 0$.

Norm: $||\psi_E^+||^2 = \frac{1}{E}\langle\psi_E^-|Q^\dagger Q|\psi_E^-\rangle = \frac{1}{E}\langle\psi_E^-|Q^\dagger Q + QQ^\dagger|\psi_E^-\rangle = \frac{1}{E}\langle\psi_E^-|H|\psi_E^-\rangle = 1$

Corollary: The spectra of the two SUSY partner Hamiltonians H_+ and H_- are identical away from zero:

$$\text{spec}\,(H_+)\backslash\{0\} = \text{spec}\,(H_-)\backslash\{0\}.$$

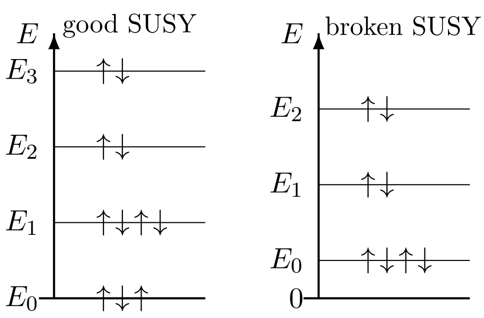
We say, Hamiltonians H_+ and H_- are *essentially isospectral.*
That is, the strictly positive eigenvalues of the SUSY partner Hamiltonians $H_\pm$ coincide.

Remarks:

- Let $|\phi_E^\pm\rangle \in \mathcal{H}^\pm$ with $E > 0$, then

$$\boxed{|\phi_E^-\rangle = \frac{1}{\sqrt{E}}\,A^\dagger|\phi_E^+\rangle, \qquad |\phi_E^+\rangle = \frac{1}{\sqrt{E}}\,A|\phi_E^-\rangle.}$$

- Spectral Properties of $N=2$ SUSY QM



Symbolic notation: $\qquad |\psi_E^+\rangle = \begin{pmatrix} |\phi_E^+\rangle \\ 0 \end{pmatrix} = |E,\uparrow\rangle \qquad |\psi_E^-\rangle = \begin{pmatrix} 0 \\ |\phi_E^-\rangle \end{pmatrix} = |E,\downarrow\rangle$

- Requirement for unbroken SUSY: $\quad \exists\ |\psi_0\rangle$ such that $Q|\psi_0\rangle = 0$ or $Q^\dagger|\psi_0\rangle = 0$

For negative ground state: $|\psi_0^-\rangle = \begin{pmatrix} 0 \\ |\phi_0^-\rangle \end{pmatrix} \qquad \Longrightarrow \qquad A|\phi_0^-\rangle = 0$

For positive ground state: $|\psi_0^+\rangle = \begin{pmatrix} |\phi_0^+\rangle \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad A^\dagger|\phi_0^+\rangle = 0$

2.3.4 The Witten Index

Definition: Let us denote by $n_\pm$ the number of zero modes (zero eigenvalues) of $H_\pm$ in the subspace $\mathcal{H}^\pm$. For finite n_+ and n_- the quantity

$$\boxed{\Delta := n_- - n_+} \qquad (2)$$

is called the *Witten index.*

Remarks:

- $\Delta \neq 0 \quad \Longrightarrow \quad$ SUSY is unbroken as at least one, n_+ or n_-, is non-zero
  $\Delta = 0 \quad \Longrightarrow \quad$ SUSY can be broken ($n_+ = n_- = 0$) or unbroken ($n_+ = n_- \neq 0$)

- Relation to Fredholm index of A, which is defined by

$$\begin{aligned} \operatorname{ind} A &:= \dim\ker A - \dim\ker A^\dagger \\ &= \dim\ker A^\dagger A - \dim\ker AA^\dagger \\ &= \dim\ker H_- - \dim\ker H_+ \\ &= n_- - n_+ \\ &= \Delta \end{aligned}$$

- Connection with Witten parity:

  Formally: $\Delta = \operatorname{Tr}(-W) = \operatorname{Tr}_{\mathcal{H}_-}(1) - \operatorname{Tr}_{\mathcal{H}_+}(1) = \dim\mathcal{H}_- - \dim\mathcal{H}_+ = n_- - n_+$

  Cancelation of the $E > 0$ contributions due to SUSY degeneracy!

  Regularised indices:

$$\begin{aligned} \bar\Delta(\beta) &:= \operatorname{Tr}\left(-W\mathrm{e}^{-\beta H}\right) = \operatorname{Tr}_-\left(\mathrm{e}^{-\beta A^\dagger A}\right) - \operatorname{Tr}_+\left(\mathrm{e}^{-\beta AA^\dagger}\right), && \beta > 0 \\ \widehat\Delta(z) &:= \operatorname{Tr}\left(-W\,\tfrac{z}{H-z}\right) = \operatorname{Tr}_-\left(\tfrac{z}{A^\dagger A-z}\right) - \operatorname{Tr}_+\left(\tfrac{z}{AA^\dagger-z}\right), && z < 0 \\ \widetilde\Delta(\varepsilon) &:= \operatorname{Tr}\Big(-W\Theta(\varepsilon-H)\Big) = \operatorname{Tr}_-\Big(\Theta(\varepsilon-A^\dagger A)\Big) - \operatorname{Tr}_+\Big(\Theta(\varepsilon-AA^\dagger)\Big), && \varepsilon > 0 \end{aligned}$$

  For purely discrete spectrum and finite $n_\pm$ follows $\Delta = \bar\Delta(\beta) = \widehat\Delta(z) = \widetilde\Delta(\varepsilon)$.
  Otherwise on defines

$$\Delta := \lim_{\beta\to\infty}\bar\Delta(\beta) \qquad \text{or} \qquad \Delta := \lim_{z\uparrow 0}\widehat\Delta(z) \qquad \text{or} \qquad \Delta := \lim_{\varepsilon\downarrow 0}\widetilde\Delta(\varepsilon)$$

  whenever the right-hand-side is well defined.
  Problems arise when $n_\pm = \infty$ and/or continuous spectrum (see Pauli paramagnetism later)

- Partition functions and internal energy:

  Let $\qquad Z_\pm(\beta) := \operatorname{Tr}\mathrm{e}^{-\beta H_\pm} \qquad$ and $\qquad U_\pm(\beta) := -\partial_\beta \ln Z_\pm(\beta) \qquad$ then

$$Z_-(\beta) = \Delta + Z_+(\beta) \qquad \text{and} \qquad U_-(\beta)Z_-(\beta) = U_+(\beta)Z_+(\beta).$$

  if A is Fredholm, i.e. $\operatorname{ind} A$ is well-defined.

- The Witten index is a topological invariant, that is, it is
  NOT sensitive against smooth variations of parameters in the theory
  $\Longrightarrow \qquad$ Witten model next section

Summary of Section 2

$N = 2$ SUSY QM with Witten parity:

System $\{H, Q, Q^\dagger, W; \mathcal{H}\}$ obeying

$$\boxed{\begin{array}{ccc} \{Q, Q^\dagger\} = H\,, & Q^2 = 0 = (Q^\dagger)^2\,, & W^2 = 1 \\ [W, H] = 0\,, & \{W, Q\} = 0 = \{W, Q^\dagger\}\,, & W = W^\dagger \end{array}}$$

Matrix representation:

Grading of Hilbert space $\quad \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \quad$ into Witten parity eigen-subspaces of

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix}, \qquad \begin{array}{l} A : \mathcal{H}^- \to \mathcal{H}^+ \\ A^\dagger : \mathcal{H}^+ \to \mathcal{H}^- \end{array}$$

Eigenstates:

$$W|\psi_E^\pm\rangle = \pm|\psi_E^\pm\rangle$$

$$H|\psi_E^\pm\rangle = E|\psi_E^\pm\rangle \qquad E \geq 0$$

SUSY transformations: Eigenvalue $E > 0$ is pairwise degenerate

$$\begin{aligned} |\psi_E^-\rangle &= \frac{1}{\sqrt{E}}\, Q^\dagger|\psi_E^+\rangle \\ |\psi_E^+\rangle &= \frac{1}{\sqrt{E}}\, Q|\psi_E^-\rangle \end{aligned} \qquad \text{modulo phase factors}$$

SUSY unbroken: $\quad E = 0$ is eigenvalue of H, $\quad$ no SUSY transformation for ground state(s)

SUSY broken: H has only strictly positive eigenvalues $E > 0$, E is pairwise degenerate