Supersymmetric Quantum Mechanics

Lecture Notes

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Preliminaries

Dates:
Six Mondays 17.04.23, 24.04.23, 08.05.23, 15.05.23, 22.05.23, 29.05.23, 05.06.23 (Test?)
Lecture 9 - 12, Tutorial 13 - 15, Homework Problems
Script and other details are available at
`https://www.eso.org/~gjunker/VorlesungSS2023.html`

Literature:

- Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer-Verlag) 1st edition
- Kalka H and Soff G 1997 *Supersymmetrie* (Stuttgart: Teubner)
- Cooper F, Khare A and Sukhatme U 2001 *Supersymmetry in Quantum Mechanics* (Singapore: World Scientific)
- Bagchi B 2001 *Supersymmetry in Quantum and Classical Mechanics* (Boca Raton: Chapman & Hall/CRC)
- Gangopadhyaya A, Mallow J V and Rasinariu C 2011 *Supersymmetric Quantum Mechanics: An Introduction* (Singapore: World Scientific)
- Junker G 2019, *Supersymmetric Methods in Quantum, Statistical and Solid State Physics* (Bristol: IOP) $\Longrightarrow$ "The Book"
- ...

Supersymmetric Quantum Mechanics:

$$\text{SUSY QM} = \text{QM} + \text{Supercharges}$$

Supercharges are conserved quantities obeying a SUSY algebra

Aim of lecture:
Supersymmetry (SUSY) as an algebraic tool with many applications in theoretical and mathematical physics and beyond.

1 Historical Background

SUSY idea originates in quantum field theory (gauge theories)

• Structure:

Space – Time Sym. (Poincare Algebra)	Internal (Gauge) Sym. (Lie Algebra)
Matter Fields (Fermions)	Gauge Fields (Bosons)

• SUSY idea: Unify space-time and internal symmetries

$\Longrightarrow$ Unification of Fermions and Bosons

NoGo-Theorem of Coleman and Mandula
Within the context of Lie algebras NOT possible

$\Longrightarrow$ Super (or graded) Lie algebras close under

$$\text{Commutator} \qquad [A, B] := AB - BA$$

and

$$\text{Anticommutator} \qquad \{A, B\} := AB + BA$$

• 1976: H. Nicolai invented SUSY QM as $(0 + 1)$-dim. QFT

• 1981: E. Witten introduced a simple QM model (Witten model)
$\Longrightarrow$ popularization

• More background is given in "The Book"

Content

- Supersymmetric Quantum Mechanics (definitions and properties)
- The Witten Model (non-relativistic SUSY QM)
- Darboux Method (construct problem with known solution)
- Classical Field in $(1 + 1)$ Dimensions (SUSY in classical systems)
- Supersymmetry in Stochastic Processes (SUSY in classical stochastic systems)
- Supersymmetry and Pauli Hamiltonians (Aharonov-Casher; Pauli paramagnetism)
- Supersymmetry and Dirac Hamiltonians (SUSY in rel. QM systems; Graphene)

2 Supersymmetric Quantum Mechanics

2.1 Definitions

Assumptions:

Hilbert space:	$\mathcal{H}$	
Hamiltonian:	$H = H^\dagger$	
Observables:	$Q_i = Q_i^\dagger,$	$i = 1, 2, 3 \dots, N$

Definition 2.1: A quantum system characterised by the set $\{H, Q_1, \dots, Q_N; \mathcal{H}\}$, is called *supersymmetric* if the following anticommutation relation is valid for all $i, j = 1, 2, \dots, N$:

$$\boxed{\{Q_i, Q_j\} = H\delta_{ij},} \tag{1}$$

where δ_{ij} denotes Kronecker's delta symbol. The self-adjoint operators Q_i are called *supercharges* and the Hamiltonian H is called *SUSY Hamiltonian.* The symmetry described by the *superalgebra* (1) is called *N-extended supersymmetry.*

Remarks:

- $H = 2Q_1^2 = 2Q_2^2 = \cdots = 2Q_N^2 = \dfrac{2}{N}\displaystyle\sum_{i=1}^{N} Q_i^2 \geq 0$

  no negative energy eigenvalues

  $Q_i = \sqrt{H/2}$ square root of Hamiltonian

- $[H, Q_i] = 0$ *supercharges* Q_i are constants of motion if $\dfrac{\partial Q_i}{\partial t} = 0$

- For $N \geq 2$ we may introduce complex supercharges

  $$\tilde{Q}_k := \frac{1}{\sqrt{2}}\left[Q_{2k-1} + \mathrm{i}Q_{2k}\right]$$

  $$\boxed{\left\{\widetilde{Q}_k, \widetilde{Q}_l^\dagger\right\} = H\delta_{kl}\,, \qquad \widetilde{Q}_k^2 = 0 = \left(\widetilde{Q}_k^\dagger\right)^2}$$

  Show that $\left\{\widetilde{Q}_k, \widetilde{Q}_l\right\} = 0$ for all k, l.

Let $E_0 := \inf \operatorname{spec} H \geq 0$ be ground state energy of H with

$$H|\psi_0^j\rangle = E_0|\psi_0^j\rangle\,, \qquad j = 1, 2, 3, \dots, g \qquad (g = \text{degeneracy of } E_0)$$

Definition:

$$\boxed{\begin{array}{ll} \text{SUSY unbroken} : & \iff E_0 = 0 \\ \text{SUSY broken} : & \iff E_0 > 0 \end{array}}$$

Remarks: $E_0 = \langle\psi_0^j|H|\psi_0^j\rangle = \dfrac{2}{N}\displaystyle\sum_{i=1}^{N}\langle\psi_0^j|Q_i^2|\psi_0^j\rangle = \dfrac{2}{N}\displaystyle\sum_{i=1}^{N}||Q_i|\psi_0^j\rangle||^2$

$E_0 = 0 \iff Q_i|\psi_0^j\rangle = 0$ for all (i, j)

$E_0 > 0 \iff \exists$ pair (i, j) such that $Q_i|\psi_0^j\rangle \neq 0$

ground state is NOT invariant under SUSY transformations and SUSY is broken

2.2 The Supersymmetric Harmonic Oscillator

Consider 1-dim. quantum particle with spin $\frac{1}{2}$ and unit mass $m = 1$

Hilbert space: $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$

"Bosonic" degree of freedom: $a := \frac{1}{\sqrt{2}}(\partial_x + x) \quad \Longrightarrow \quad [a, a^\dagger] = 1$

"Fermionic" degree of freedom: $b := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \Longrightarrow \quad \{b, b^\dagger\} = 1\,, \quad b^2 = 0 = (b^\dagger)^2$

Complex Supercharge: Use no longer tilde $\tilde{Q} \equiv Q$

$$Q := a \otimes b^\dagger = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \qquad Q^\dagger = a^\dagger \otimes b = \begin{pmatrix} 0 & 0 \\ a^\dagger & 0 \end{pmatrix}$$

SUSY Hamiltonian:

$$\begin{aligned} H &:= \{Q, Q^\dagger\} = a^\dagger a + b^\dagger b \\ &= \tfrac{1}{2}(-\partial_x^2 + x^2 - 1) \otimes 1 + 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \tfrac{1}{2}(-\partial_x^2 + x^2) \otimes 1 + 1 \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \end{aligned}$$

Spectral properties of H:

- Eigenstates

$$|n, \downarrow\rangle := |n\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad |n, \uparrow\rangle := |n\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n \in \mathbb{N}_0\,,$$

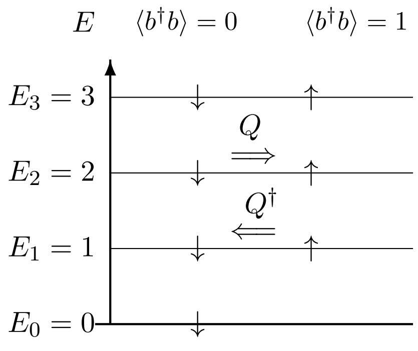
  where

$$a|n\rangle = \sqrt{n}|n-1\rangle\,, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

  and

$$b|n,\uparrow\rangle = |n,\downarrow\rangle\,, \qquad b|n,\downarrow\rangle = 0\,, \qquad b^\dagger|n,\downarrow\rangle = |n,\uparrow\rangle\,, \qquad b^\dagger|n,\uparrow\rangle = 0\,.$$

- Eigenvalues: $\langle a^\dagger a\rangle = n\,, \quad n = 0, 1, 2, 3, \ldots$



- SUSY unbroken as $E_0 = 0$
- $E > 0$ pairwise degenerate

SUSY Transformations:

$$Q|n,\downarrow\rangle = \sqrt{n}|n-1,\uparrow\rangle\,, \qquad Q|n,\uparrow\rangle = 0\,,$$

$$Q^\dagger|n,\uparrow\rangle = \sqrt{n+1}|n+1,\downarrow\rangle\,, \qquad Q^\dagger|n,\downarrow\rangle = 0\,.$$

Q and $Q^\dagger$ transform between spin-down and spin-up state with SAME energy eigenvalue. Is generic property for all $N \geq 2$ SUSY QM systems as there exists a Witten parity operator.

2.3 Properties of $N = 2$ SUSY QM

Consider $N = 2$ SUSY QM: $\{H, Q_1, Q_2; \mathcal{H}\}$

Recall: $Q_1 Q_2 = -Q_2 Q_1, \qquad H = 2Q_1^2 = 2Q_2^2 = Q_1^2 + Q_2^2$

Complex Supercharge: $Q := \frac{1}{\sqrt{2}}(Q_1 + \mathrm{i}Q_2), \qquad Q^\dagger = \frac{1}{\sqrt{2}}(Q_1 - \mathrm{i}Q_2)$

SUSY algebra:

$$\boxed{Q^2 = 0 = (Q^\dagger)^2\,, \qquad \{Q, Q^\dagger\} = H}$$

2.3.1 The Witten parity

Let us assume there exists a self-adjoint operator W such that

$$\boxed{[W, H] = 0, \qquad \{W, Q\} = 0 = \{W, Q^\dagger\}, \qquad W^2 = \mathbf{1}.}$$

Definition: A self-adjoint operator W which obeys above relations is called *Witten parity* or *Witten operator*. The quantum system $\{H, Q, Q^\dagger, W; \mathcal{H}\}$ will be called a *supersymmetric quantum system with Witten parity.*

Remarks: See Tutorial Exercise 2 and 3

- $\operatorname{spec} W = \{-1, +1\}$ non-trivial unitary involution on $\mathcal{H}$
  $[Q, H] = 0 = [Q^\dagger, H]$ constant of motion
- For $N \geq 2$ formal construction on $\mathcal{H}\backslash\ker(H)$ via

$$\boxed{W := \frac{2}{H} QQ^\dagger - \mathbf{1} = \frac{1}{\mathrm{i}H}\,[Q_1, Q_2] = \frac{[Q, Q^\dagger]}{\{Q, Q^\dagger\}}}$$

- "Fermionic" annihilation operator

$$b := Q^\dagger/\sqrt{H} \qquad \text{on} \qquad \mathcal{H}\backslash\ker(H)$$

  obeying the relations

$$\{b, b^\dagger\} = \mathbf{1}, \qquad b^2 = 0 = \left(b^\dagger\right)^2.$$

- "Fermion" number operator

$$\mathcal{F} := b^\dagger b = QQ^\dagger/H = \mathcal{F}^\dagger = \mathcal{F}^2$$

  obeys the algebra

$$[\mathcal{F}, H] = 0\,, \qquad [\mathcal{F}, Q] = Q\,, \qquad [\mathcal{F}, Q^\dagger] = -Q^\dagger\,,$$

  and is related to the Witten parity by

$$W = 2\mathcal{F} - \mathbf{1} = (-\mathbf{1})^{\mathcal{F}+1}\,.$$

2.3.2 Witten parity subspaces

Definition: Let $P^\pm := \frac{1}{2}(1 \pm W)$ be the orthogonal projection of $\mathcal{H}$ onto the eigenspace of the Witten operator with eigenvalue ± 1, respectively.
The subspace

$$\mathcal{H}^\pm := P^\pm \mathcal{H} P^\pm = \{|\psi\rangle \in \mathcal{H} \,:\, W|\psi\rangle = \pm|\psi\rangle\}$$

is called space of *positive* ($\mathcal{H}^+$) and *negative* ($\mathcal{H}^-$) Witten parity, respectively.

Remarks:

- Projectors:

  $P^\pm P^\pm = \frac{1}{4}(1 \pm W)(1 \pm W) = \frac{1}{4}(1 \pm 2W + W^2) = \frac{1}{2}(1 \pm W) = P^\pm$ projector

  $P^\pm P^\mp = \frac{1}{4}(1 \pm W)(1 \mp W) = \frac{1}{4}(1 - W^2) = 0$ orthogonal

  $P^+ + P^- = 1$ complete

  $\Longrightarrow \quad \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ grading of $\mathcal{H}$ induced by W.

- Matrix representation

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad P^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$|\psi^+\rangle = \begin{pmatrix} |\phi^+\rangle \\ 0 \end{pmatrix}, \qquad |\psi^-\rangle = \begin{pmatrix} 0 \\ |\phi^-\rangle \end{pmatrix}, \qquad |\phi^\pm\rangle \in \mathcal{H}^\pm$$

- Supercharges: $\{Q, W\} = 0$

  $\Longrightarrow \quad \pm Q|\psi^\pm\rangle = QW|\psi^\pm\rangle = -WQ|\psi^\pm\rangle \quad \Longrightarrow \quad WQ|\psi^\pm\rangle = \mp Q|\psi^\pm\rangle$

  Hence $\quad Q|\psi^\pm\rangle \in \mathcal{H}^\mp \quad$ or $\quad Q|\psi^\pm\rangle = 0$

  Similar $\quad Q^\dagger|\psi^\pm\rangle \in \mathcal{H}^\mp \quad$ or $\quad Q^\dagger|\psi^\pm\rangle = 0$

  Q and $Q^\dagger$ transform between $\mathcal{H}^+$ and $\mathcal{H}^-$ $\quad \Longrightarrow \quad$ *SUSY transformations*

  Hence $Q\mathcal{H}^- \subset \mathcal{H}^+, \quad Q^\dagger\mathcal{H}^+ \subset \mathcal{H}^-$

- Without loss of generality (see Tutorial Exercise 4):

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix},$$

  with $\quad A : \mathcal{H}^- \to \mathcal{H}^+ \quad$ and $\quad A^\dagger : \mathcal{H}^+ \to \mathcal{H}^-$

  Observe $\quad Q^\dagger\mathcal{H}^- = 0 = Q\mathcal{H}^+$

- SUSY partner Hamiltonians:

$$H = Q^\dagger Q + QQ^\dagger = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

  with *SUSY partner Hamiltonians*

$$H_+ := AA^\dagger \geq 0, \qquad H_- := A^\dagger A \geq 0.$$

- Even and odd operators:
  An arbitrary operator O acting on $\mathcal{H}$ can be decomposed into its diagonal (even) part O_e and its off-diagonal (odd) part O_o. That is, $O = O_\mathrm{e} + O_\mathrm{o}$ with

$$[W, O_\mathrm{e}] = 0, \qquad \{W, O_\mathrm{o}\} = 0.$$

  In general

$$O = \begin{pmatrix} O_{++} & O_{+-} \\ O_{-+} & O_{--} \end{pmatrix}$$

  with O_{++} and O_{--} forming the even part and O_{+-} and O_{-+} the odd part of O.
  In particular, the SUSY Hamiltonian H is an even operator, whereas the supercharges Q and $Q^\dagger$ are odd operators.

2.3.3 SUSY Transformations

Definition: Eigenstates of W are called *positive* and *negative (Witten-) parity states*, respectively. They are denoted by $|\psi^{\pm}\rangle$:

$$W|\psi^{\pm}\rangle = \pm|\psi^{\pm}\rangle\,, \qquad |\psi^{\pm}\rangle \in \mathcal{H}^{\pm}\,.$$

For simplicity we will call them also positive and negative states.

Proposition: To each positive (negative) eigenstate $|\psi_E^+\rangle$ ($|\psi_E^-\rangle$) of the SUSY Hamiltonian H with eigenvalue $E > 0$ there exists a negative (positive) eigenstate of H with the same eigenvalue. These eigenstates are related by the *SUSY transformations*

$$\boxed{|\psi_E^-\rangle = \frac{1}{\sqrt{E}}\, Q^\dagger|\psi_E^+\rangle, \qquad |\psi_E^+\rangle = \frac{1}{\sqrt{E}}\, Q|\psi_E^-\rangle,}$$

where

$$W|\psi_E^{\pm}\rangle = \pm|\psi_E^{\pm}\rangle \qquad \text{and} \qquad H|\psi_E^{\pm}\rangle = E|\psi_E^{\pm}\rangle\,.$$

Proof: As $[W,H] = 0 \qquad \Longrightarrow \qquad$ common eigenbasis

Let $H|\psi_E^-\rangle = E|\psi_E^-\rangle \qquad \Longrightarrow \qquad HQ|\psi_E^-\rangle = QH|\psi_E^-\rangle = EQ|\psi_E^-\rangle \in \mathcal{H}^+$.

$\Longrightarrow \qquad |\psi_E^+\rangle := \frac{1}{\sqrt{E}}Q|\psi_E^-\rangle$ is positive eigenstate of H for the same eigenvalue $E > 0$.

Norm: $||\psi_E^+||^2 = \frac{1}{E}\langle\psi_E^-|Q^\dagger Q|\psi_E^-\rangle = \frac{1}{E}\langle\psi_E^-|Q^\dagger Q + QQ^\dagger|\psi_E^-\rangle = \frac{1}{E}\langle\psi_E^-|H|\psi_E^-\rangle = 1$

Corollary: The spectra of the two SUSY partner Hamiltonians H_+ and H_- are identical away from zero:

$$\text{spec}\,(H_+)\backslash\{0\} = \text{spec}\,(H_-)\backslash\{0\}.$$

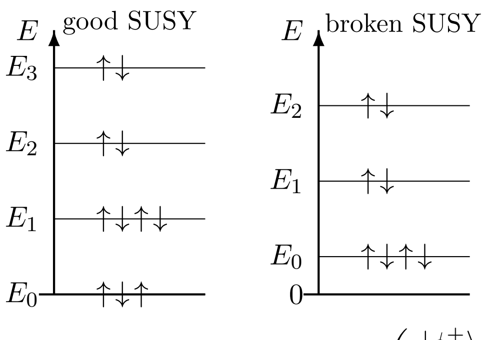
We say, Hamiltonians H_+ and H_- are *essentially isospectral.*
That is, the strictly positive eigenvalues of the SUSY partner Hamiltonians $H_\pm$ coincide.

Remarks:

- Let $|\phi_E^{\pm}\rangle \in \mathcal{H}^{\pm}$ with $E > 0$, then

$$\boxed{|\phi_E^-\rangle = \frac{1}{\sqrt{E}}\, A^\dagger|\phi_E^+\rangle, \qquad |\phi_E^+\rangle = \frac{1}{\sqrt{E}}\, A|\phi_E^-\rangle.}$$

- Spectral Properties of $N = 2$ SUSY QM



Symbolic notation: $\qquad |\psi_E^+\rangle = \begin{pmatrix} |\phi_E^+\rangle \\ 0 \end{pmatrix} = |E,\uparrow\rangle \qquad |\psi_E^-\rangle = \begin{pmatrix} 0 \\ |\phi_E^-\rangle \end{pmatrix} = |E,\downarrow\rangle$

- Requirement for unbroken SUSY: $\quad \exists\ |\psi_0\rangle$ such that $Q|\psi_0\rangle = 0$ or $Q^\dagger|\psi_0\rangle = 0$

For negative ground state: $|\psi_0^-\rangle = \begin{pmatrix} 0 \\ |\phi_0^-\rangle \end{pmatrix} \qquad \Longrightarrow \qquad A|\phi_0^-\rangle = 0$

For positive ground state: $|\psi_0^+\rangle = \begin{pmatrix} |\phi_0^+\rangle \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad A^\dagger|\phi_0^+\rangle = 0$

2.3.4 The Witten Index

Definition: Let us denote by $n_\pm$ the number of zero modes (zero eigenvalues) of $H_\pm$ in the subspace $\mathcal{H}^\pm$. For finite n_+ and n_- the quantity

$$\boxed{\Delta := n_- - n_+} \tag{2}$$

is called the *Witten index*.

Remarks:

- $\Delta \neq 0 \quad \Longrightarrow \quad$ SUSY is unbroken as at least one, n_+ or n_-, is non-zero
  $\Delta = 0 \quad \Longrightarrow \quad$ SUSY can be broken ($n_+ = n_- = 0$) or unbroken ($n_+ = n_- \neq 0$)

- Relation to Fredholm index of A, which is defined by

$$\begin{aligned} \operatorname{ind} A &:= \dim\ker A - \dim\ker A^\dagger \\ &= \dim\ker A^\dagger A - \dim\ker AA^\dagger \\ &= \dim\ker H_- - \dim\ker H_+ \\ &= n_- - n_+ \\ &= \Delta \end{aligned}$$

- Connection with Witten parity:

  Formally: $\Delta = \operatorname{Tr}(-W) = \operatorname{Tr}_{\mathcal{H}_-}(1) - \operatorname{Tr}_{\mathcal{H}_+}(1) = \dim\mathcal{H}_- - \dim\mathcal{H}_+ = n_- - n_+$

  Cancelation of the $E > 0$ contributions due to SUSY degeneracy!

  Regularised indices:

$$\begin{aligned} \bar\Delta(\beta) &:= \operatorname{Tr}\left(-W e^{-\beta H}\right) = \operatorname{Tr}_-\left(e^{-\beta A^\dagger A}\right) - \operatorname{Tr}_+\left(e^{-\beta AA^\dagger}\right), && \beta > 0 \\ \widehat\Delta(z) &:= \operatorname{Tr}\left(-W\,\tfrac{z}{H-z}\right) = \operatorname{Tr}_-\left(\tfrac{z}{A^\dagger A - z}\right) - \operatorname{Tr}_+\left(\tfrac{z}{AA^\dagger - z}\right), && z < 0 \\ \widetilde\Delta(\varepsilon) &:= \operatorname{Tr}\Big(-W\Theta(\varepsilon - H)\Big) = \operatorname{Tr}_-\Big(\Theta(\varepsilon - A^\dagger A)\Big) - \operatorname{Tr}_+\Big(\Theta(\varepsilon - AA^\dagger)\Big), && \varepsilon > 0 \end{aligned}$$

  For purely discrete spectrum and finite $n_\pm$ follows $\Delta = \bar\Delta(\beta) = \widehat\Delta(z) = \widetilde\Delta(\varepsilon)$.
  Otherwise on defines

$$\Delta := \lim_{\beta\to\infty}\bar\Delta(\beta) \qquad \text{or} \qquad \Delta := \lim_{z\uparrow 0}\widehat\Delta(z) \qquad \text{or} \qquad \Delta := \lim_{\varepsilon\downarrow 0}\widetilde\Delta(\varepsilon)$$

  whenever the right-hand-side is well defined.
  Problems arise when $n_\pm = \infty$ and/or continuous spectrum (see Pauli paramagnetism later)

- Partition functions and internal energy:

  Let $\quad Z_\pm(\beta) := \operatorname{Tr} e^{-\beta H_\pm} \quad$ and $\quad U_\pm(\beta) := -\partial_\beta \ln Z_\pm(\beta) \quad$ then

$$Z_-(\beta) = \Delta + Z_+(\beta) \qquad \text{and} \qquad U_-(\beta)Z_-(\beta) = U_+(\beta)Z_+(\beta).$$

  if A is Fredholm, i.e. $\operatorname{ind} A$ is well-defined.

- The Witten index is a topological invariant, that is, it is
  NOT sensitive against smooth variations of parameters in the theory
  $\Longrightarrow \quad$ Witten model next section

Summary of Section 2

$N = 2$ SUSY QM with Witten parity:

System $\{H, Q, Q^\dagger, W; \mathcal{H}\}$ obeying

$$\boxed{\begin{array}{ccc} \{Q, Q^\dagger\} = H\,, & Q^2 = 0 = (Q^\dagger)^2\,, & W^2 = 1 \\ [W, H] = 0\,, & \{W, Q\} = 0 = \{W, Q^\dagger\}\,, & W = W^\dagger \end{array}}$$

Matrix representation:

Grading of Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ into Witten parity eigen-subspaces of

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix}, \qquad \begin{array}{l} A : \mathcal{H}^- \to \mathcal{H}^+ \\ A^\dagger : \mathcal{H}^+ \to \mathcal{H}^- \end{array}$$

Eigenstates:

$$W|\psi_E^\pm\rangle = \pm|\psi_E^\pm\rangle$$
$$H|\psi_E^\pm\rangle = E|\psi_E^\pm\rangle \qquad E \geq 0$$

SUSY transformations: Eigenvalue $E > 0$ is pairwise degenerate

$$|\psi_E^-\rangle = \frac{1}{\sqrt{E}}\, Q^\dagger|\psi_E^+\rangle$$
$$|\psi_E^+\rangle = \frac{1}{\sqrt{E}}\, Q|\psi_E^-\rangle \qquad \text{modulo phase factors}$$

SUSY unbroken: $E = 0$ is eigenvalue of H, no SUSY transformation for ground state(s)

SUSY broken: H has only strictly positive eigenvalues $E > 0$, E is pairwise degenerate

3 The Witten Model

Simple 1-dim. model with $N = 2$ SUSY for quantum particle with mass $m > 0$.

Definitions:

- Hilbert space: $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathbb{C}^2 = \mathcal{H}^+ \oplus \mathcal{H}^-, \quad \mathcal{H}^\pm = L^2(\mathbb{R})$
- Super charge: $Q := \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ with $A := \frac{\mathrm{i}}{\sqrt{2m}} P + \Phi(x) = \frac{\hbar}{\sqrt{2m}} \partial_x + \Phi(x)$
- *SUSY potential*: $\Phi : x \mapsto \Phi(x)$ piecewise continous diff. on $\mathbb{R}$
- Hamiltonian: $H := \{Q, Q^\dagger\} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix}$
- SUSY partner Hamiltonians:

$$\boxed{H_\pm = \frac{P^2}{2m} + \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}} \Phi'(x) = \frac{P^2}{2m} + V_\pm(x)}$$

- SUSY Partner potentials: $V_\pm(x) := \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}} \Phi'(x)$

Remarks:

- Partner Hamiltonians are standard 1-dim. Schrödinger Hamiltonians with special form of the potential
- $\Phi(x) = \sqrt{\frac{m}{2}}\, \omega\, x$ represents the supersymmetric harmonic oscillator
- Conf. space may be replaced by a subspace $\mathcal{M} \subset \mathbb{R}$ with suitable boundary conditions
  $\mathcal{M} = \mathbb{R}^+$ positive half line, radial problems
  $\mathcal{M} = [a, b]$ finite interval, particle in a box

3.1 Ground state for unbroken SUSY

SUSY unbroken $\Longrightarrow$ there exists a $|\phi_0^+\rangle$ and/or $|\phi_0^-\rangle$ such that

$$H_+|\phi_0^+\rangle = 0 \qquad \text{and/or} \qquad H_-|\phi_0^-\rangle = 0$$

$$\Longleftrightarrow \qquad A^\dagger|\phi_0^+\rangle = 0 \qquad \text{and/or} \qquad A|\phi_0^-\rangle = 0$$

$$\Longleftrightarrow \qquad \left(\mp \tfrac{\hbar}{\sqrt{2m}}\, \partial_x + \Phi(x)\right) \phi_0^\pm(x) = 0$$

$$\Longleftrightarrow \qquad \boxed{\phi_0^\pm(x) = \mathcal{N} \exp\left\{ \pm \frac{\sqrt{2m}}{\hbar} \int_{x_0}^{x} \mathrm{d}z\, \Phi(z) \right\} = \mathcal{N} \mathrm{e}^{\pm U(x)} \in \mathcal{H}^\pm}$$

with *superpotential*:

$$U(x) := \frac{\sqrt{2m}}{\hbar} \int_{x_0}^{x} \mathrm{d}z\, \Phi(z)$$

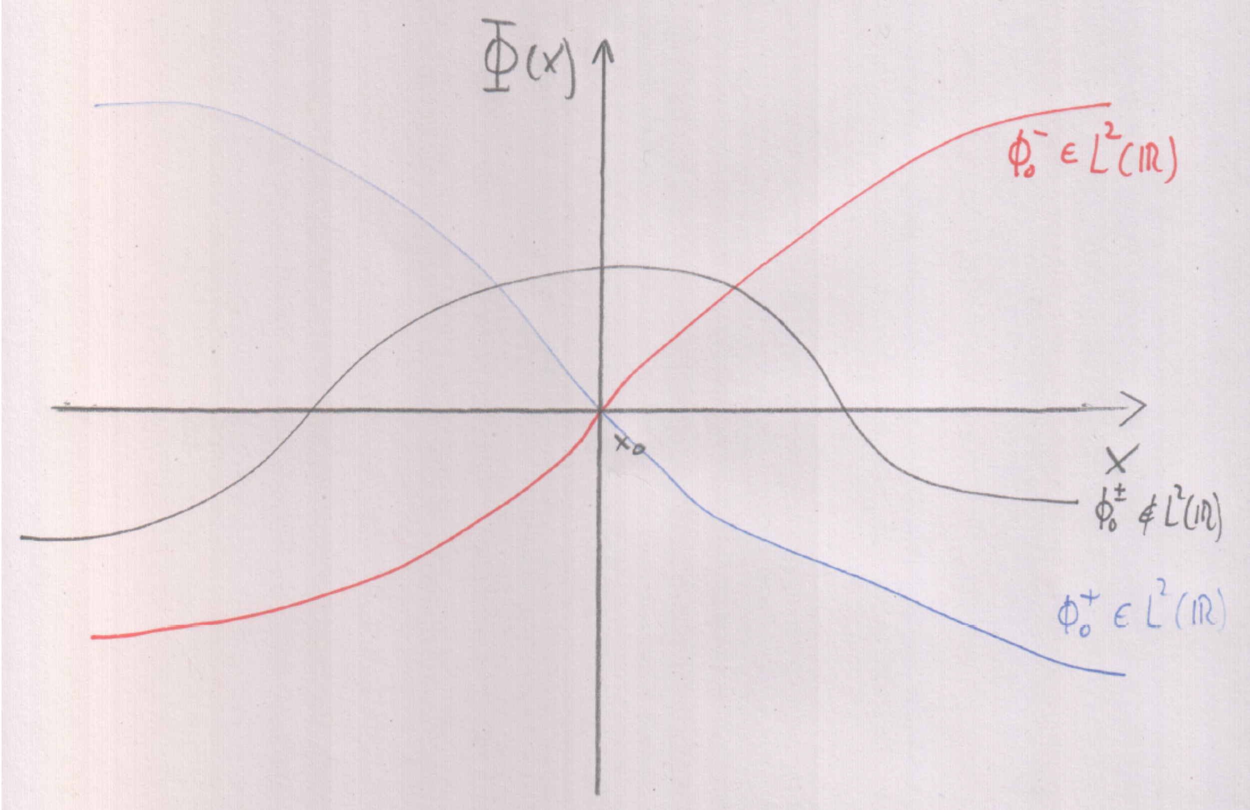
Conclusion:
Only asymptotic behaviour of SUSY potential relevant for SUSY being broken or unbroken! Let

$$\Phi_\pm := \lim_{x \to \pm\infty} \Phi(x)$$

then

$$\Delta = \frac{1}{2}(\operatorname{sgn} \Phi_+ - \operatorname{sgn} \Phi_-)$$

Witten index is *topological invariant* as it does NOT depend on the details of Φ!



Graphical discussion:

SUSY unbroken: $\Delta = +1$ $\phi_0^- \in \mathcal{H}^-$

SUSY unbroken: $\Delta = -1$ $\phi_0^+ \in \mathcal{H}^+$

SUSY broken: $\Delta = 0$

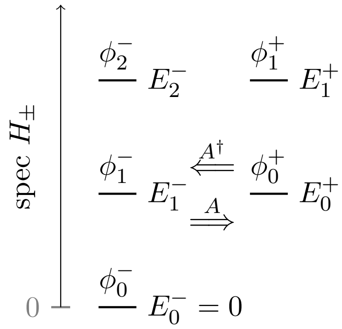
3.2 Spectral properties and SUSY transformations

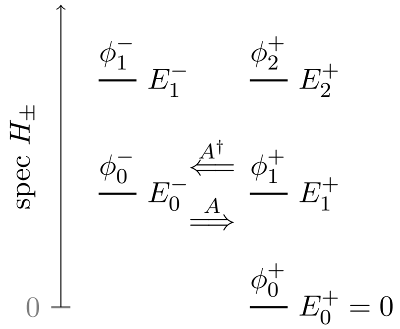
Let us assume both partner Hamiltonians have a purely discrete spectrum, that is,

$$H_\pm|\phi_n^\pm\rangle = E_n^\pm|\phi_n^\pm\rangle\,, \qquad E_n^\pm < E_{n+1}^\pm\,, \qquad n = 0,1,2,3,\ldots$$

- Good SUSY

$\boxed{\Delta = +1}$ $\boxed{\Delta = -1}$





$$E_0^- = 0\,, \qquad E_{n+1}^- = E_n^+ > 0$$
$$\phi_0^-(x) = \mathcal{N}\exp\left\{-\frac{\sqrt{2m}}{\hbar}\int \mathrm{d}x\,\Phi(x)\right\}$$
$$|\phi_{n+1}^-\rangle = \frac{1}{\sqrt{E_n^+}}\,A^\dagger|\phi_n^+\rangle$$
$$|\phi_n^+\rangle = \frac{1}{\sqrt{E_{n+1}^-}}\,A|\phi_{n+1}^-\rangle$$
$$A|\phi_0^-\rangle = 0$$

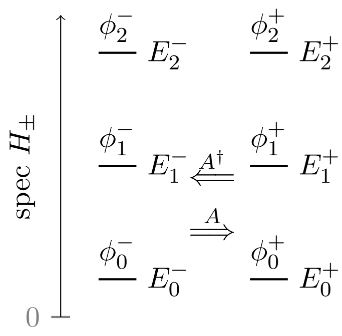
$$E_0^+ = 0\,, \qquad E_{n+1}^+ = E_n^- > 0$$
$$\phi_0^+(x) = \mathcal{N}\exp\left\{\frac{\sqrt{2m}}{\hbar}\int \mathrm{d}x\,\Phi(x)\right\}$$
$$|\phi_{n+1}^+\rangle = \frac{1}{\sqrt{E_n^-}}\,A|\phi_n^-\rangle$$
$$|\phi_n^-\rangle = \frac{1}{\sqrt{E_{n+1}^+}}\,A^\dagger|\phi_{n+1}^+\rangle$$
$$A^\dagger|\phi_0^+\rangle = 0$$

Sign convention $\Phi \leftrightarrow -\Phi$ such that for good SUSY $\Delta = +1$

$$\Longrightarrow \qquad \mathrm{sgn}\,\Phi_- < 0 < \mathrm{sgn}\,\Phi_+$$

- Broken SUSY

$\boxed{\Delta = 0}$



$$E_n^- = E_n^+ > 0$$
$$|\phi_n^-\rangle = \frac{1}{\sqrt{E_n^+}}\,A^\dagger|\phi_n^+\rangle$$
$$|\phi_n^+\rangle = \frac{1}{\sqrt{E_n^-}}\,A|\phi_n^-\rangle q$$

Examples:

- Unbroken SUSY $\Delta = +1$

$$\Phi(x) = \frac{a\hbar}{\sqrt{2m}}\mathrm{sgn}\,(x)|x|^\alpha\,, \qquad a > 0\,, \qquad \alpha > 0$$
$$V_\pm(x) = \frac{\hbar^2 a^2}{2m}\left(|x|^{2\alpha} \pm \frac{\alpha}{a}|x|^{\alpha-1}\right)$$ [draw graphs]
$$\phi_0^-(x) = \mathcal{N}\exp\left\{-\tfrac{a}{\alpha+1}|x|^{\alpha+1}\right\}$$

- Broken SUSY $\Delta = 0$
  $\Phi(x) = \frac{a\hbar}{\sqrt{2m}} |x|^\alpha\,, \qquad a > 0\,, \qquad \alpha > 0$
  $V_\pm(x) = \frac{\hbar^2 a^2}{2m} \left(|x|^{2\alpha} \pm \frac{\alpha}{a} \mathrm{sgn}\,(x)|x|^{\alpha-1} \right)$ [draw graphs]
- More examples $\Longrightarrow$ Tutorial

3.3 Shape invariance and exact solutions

Assumption: SUSY potential depends on some parameter a, that is,

$$\Phi(\cdot, x) : a \mapsto \Phi(a, x)\,, \qquad a \in I \subseteq \mathbb{R}$$

Hence,

$$V_\pm(a, x) = \Phi^2(a, x) \pm \frac{\hbar}{\sqrt{2m}} \frac{\partial}{\partial x} \Phi(a, x).$$

Definition: The partner potentials $V_\pm(a_0, x)$ are called *shape-invariant* if they are related by

$$\boxed{V_+(a_0, x) = V_-(a_1, x) + R(a_1)\,, \qquad \forall\, x \in \mathbb{R}\,,}$$

where a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping $F : a_0 \mapsto a_1 = F(a_0)$ and the residual term $R(a_1)$ is independent of the variable x.

Example:

$$\Phi(a, x) := \frac{\hbar}{\sqrt{2m}}\, a \tanh x, \qquad a > 0.$$

$$V_\pm(a, x) = \frac{\hbar^2}{2m} \left[a^2 - \frac{a(a \mp 1)}{\cosh^2 x} \right]$$

Obviously

$$V_+(a_0, x) = V_-(a_0 - 1, x) + \frac{\hbar^2}{2m} \left[a_0^2 - (a_0 - 1)^2 \right].$$

Therefore

$$a_1 = F(a_0) = a_0 - 1, \qquad R(a_1) = \frac{\hbar^2}{2m} \left[a_0^2 - a_1^2 \right] = \frac{\hbar^2}{2m} \left[a_0^2 - (a_0 - 1)^2 \right] > 0\,,$$

and

$$\phi_0^-(a_0, x) = \frac{\mathcal{N}}{\cosh^{a_0} x}.$$

Let us assume we have a family of pairwise shape invariant potentials

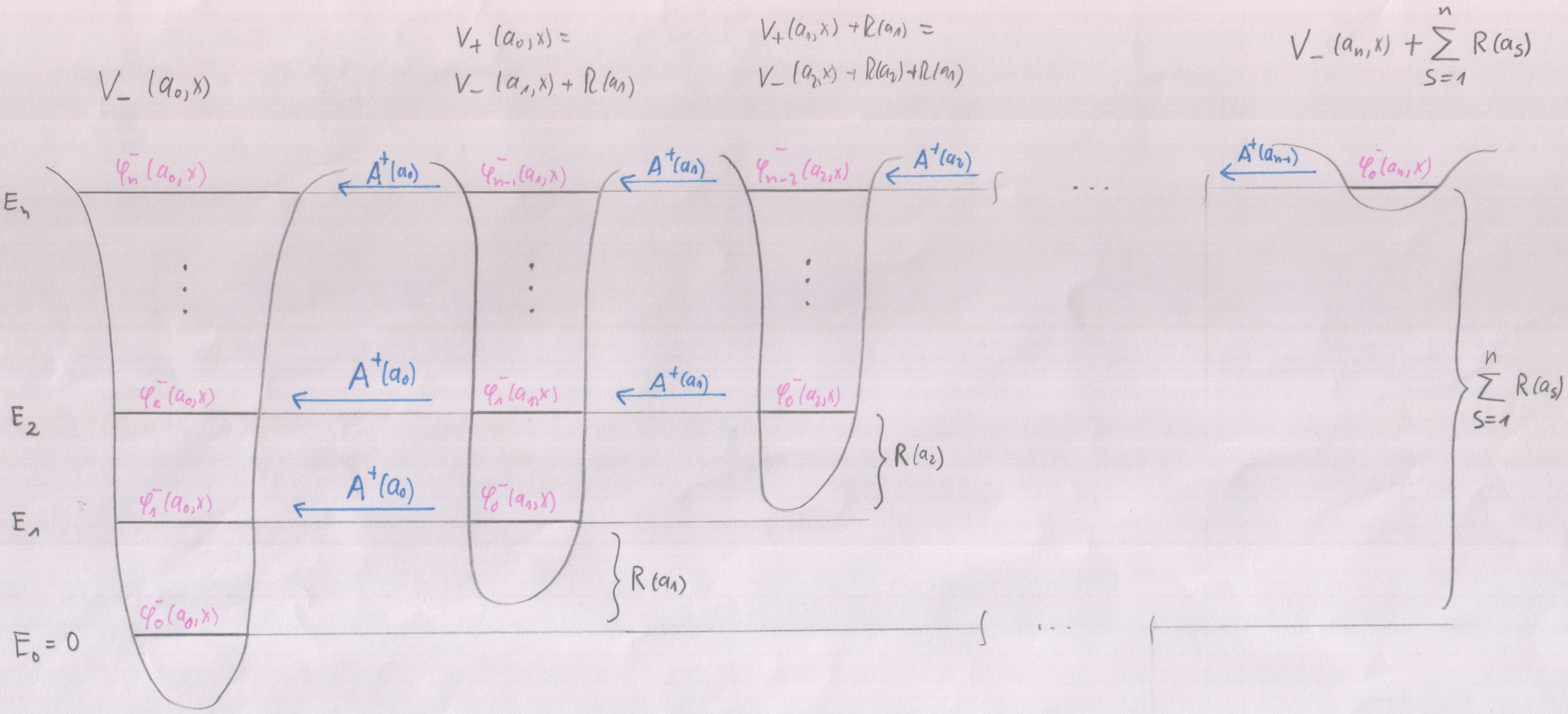
$$\{\Phi(a_s, x)\}\,, \qquad s = 0, 1, 2, \ldots, n$$

such that for all $\Delta = +1$.
Some obvious relations follow from graph on next page:

$$E_0 = 0\,, \qquad E_n = \sum_{s=1}^{n} R(a_s)$$

$$\phi_0^-(a_s, x) = \mathcal{N} \exp\left\{ -\frac{\sqrt{2m}}{\hbar} \int_0^x \mathrm{d}z\, \Phi(a_s, z) \right\}$$

$$\Phi_{n-s}^-(a_s, x) = \frac{1}{\sqrt{E_n - E_s}}\, A^\dagger(a_s) \Phi_{n-(s+1)}^-(a_s, x)$$



$$A^{+}(a_s) := -\frac{\hbar}{\sqrt{2m}}\,\partial_x + \Phi(a_s, x)$$

Conclusion: Spectral properties of $H = \frac{P^2}{2m} + V_-(a_0, x)$ are given by

$$\boxed{\begin{aligned} E_n &= \sum_{s=1}^{n} R(a_s), \\ \phi_n^-(a_0, x) &= \frac{A^\dagger(a_0)}{[E_n - E_0]^{1/2}} \cdots \frac{A^\dagger(a_{n-1})}{[E_n - E_{n-1}]^{1/2}}\, \phi_0^-(a_n, x), \\ \phi_0^-(a_n, x) &= \mathcal{N} \exp\left\{ -\frac{\sqrt{2m}}{\hbar} \int_0^x \mathrm{d}z\, \Phi(a_n, z) \right\}, \end{aligned}}$$

with

$$A^\dagger(a_s) := -\frac{\hbar}{\sqrt{2m}} \frac{\partial}{\partial x} + \Phi\,, \qquad \text{and} \qquad a_s = F(a_{s-1}(a_s, x)$$

The eigenvalue problem (discrete part) of a family of shape invariant Hamiltonians is exactly solvable

Remark:
This is not a new result due to SUSY QM but basically the well-known Schrödinger-Infeld-Hull factorization method [Rev. Mod. Phys. 23 (1951) 21]

Table of shape invariant potentials:

SUSY potential $\Phi(x)/\frac{\hbar}{\sqrt{2m}}$	config. space[a]	parameter range for good SUSY[b]	partner potentials $V_\pm(x)/\frac{\hbar^2}{2m}$
$A\tanh x + B/\cosh x$	$\mathbb{R}$	$A > 0$	$A^2 + \frac{B^2 - A(A\mp1) + B(2A\mp1)\sinh x}{\cosh^2 x}$
$A\coth x - B/\sinh x$	$\mathbb{R}^+$	$B > A > 0$	$A^2 + \frac{B^2 + A(A\mp1) - B(2A\pm1)\cosh x}{\sinh^2 x}$
$-A\cot x + B/\sin x$	$[0, \pi]$	$A > B > 0$	$-A^2 + \frac{B^2 + A(A\pm1) - B(2A\mp1)\cos x}{\sin^2 x}$
$A\tan x - B\cot x$	$[0, \pi/2]$	$A > 0,\ B > 0$[c]	$-(A+B)^2 + \frac{A(A\pm1)}{\cos^2 x} + \frac{B(B\pm1)}{\sin^2 x}$
$A\tanh x - B\coth x$	$\mathbb{R}^+$	$A > B > 0$[c]	$(A-B)^2 - \frac{A(A\mp1)}{\cosh^2 x} + \frac{B(B\pm1)}{\sinh^2 x}$
$A\tanh x + B/A$	$\mathbb{R}$	$A > B \geq 0$	$A^2 + \frac{B^2}{A^2} - \frac{A(A\mp1)}{\cosh^2 x} + 2B\tanh x$
$-A\coth x + B/A$	$\mathbb{R}^+$	$B > A > 0$	$A^2 + \frac{B^2}{A^2} + \frac{A(A\pm1)}{\sinh^2 x} - 2B\coth x$
$-A\cot x + B/A$	$[0, \pi]$	$A > 0$	$-A^2 + \frac{B^2}{A^2} + \frac{A(A\pm1)}{\sin^2 x} - 2B\cot x$
$Ax - B/x$	$\mathbb{R}^+$	$A > 0,\ B > 0$[c]	$-A(2B\mp1) + A^2x^2 + \frac{B(B\pm1)}{x^2}$
$-A/x + B/A$	$\mathbb{R}^+$	$A > 0,\ B > 0$	$\frac{B^2}{A^2} - \frac{2B}{x} + \frac{A(A\pm1)}{x^2}$
$-A\mathrm{e}^{-x} + B$	$\mathbb{R}$	$A > 0,\ B > 0$	$B^2 + A^2\mathrm{e}^{-2x} - A(2B\mp1)\mathrm{e}^{-x}$
$Ax + B$	$\mathbb{R}$	$A > 0$	$(Ax+B)^2 \pm A$

[a] For $x \in \mathbb{R}^+$, $x \in [0, \pi/2]$, and $x \in [0, \pi]$ we impose Dirichlet boundary conditions on the wave functions at $x = 0$, $x = 0, \pi/2$, and $x = 0, \pi$, respectively.
[b] With our convention that the ground state is an eigenstate of H_-.
[c] These examples belong to class 2 of Gendensthеîn and will give rise to a broken SUSY potential if B is replaced by $-B$.

Our Example:

$$\Phi(a,x) := \frac{\hbar}{\sqrt{2m}}\, a \tanh x, \qquad a > 0$$
$$V_\pm(a_0,x) = \tfrac{\hbar^2}{2m}\left[a_0^2 - \tfrac{a_0(a_0\pm 1)}{\cosh^2 x}\right]$$
$$a_s = F(a_{s-1}) = a_{s-1} - 1 = a_0 - s$$
$$R(a_s) = \frac{\hbar^2}{2m}\left[a_{s-1}^2 - a_s^2\right],$$

SUSY ground state normalizable for $n < a_0$

$$\phi_0^-(a_n,x) = \phi_0^-(a_0-n,x) = C\cosh^{n-a_0} x$$

Eigenvalues

$$E_n = \frac{\hbar^2}{2m}\sum_{s=1}^{n}(a_{s-1}^2 - a_s^2) = \frac{\hbar^2}{2m}[a_0^2 - (a_0-n)^2], \qquad n = 0,1,2,\ldots < a_0.$$

Eigenfunctions

$$\phi_n^-(a_0,x) = C_n\left[-\partial_x + a_0\tanh x\right]\cdots\left[-\partial_x + (a_0-n+1)\tanh x\right]\cosh^{n-a_0} x$$

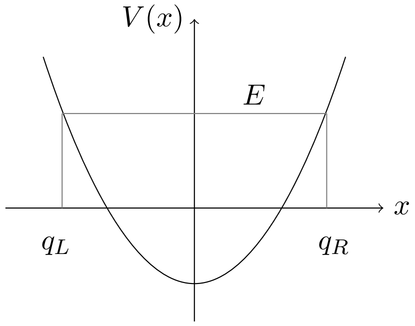
where

$$C_n := \mathcal{N}\prod_{s=0}^{n-1}\left[(a_0-s)^2 - (a_0-n)^2\right]^{-1/2}, \qquad n = 1,2,3,\ldots < a_0.$$

3.4 Quasi classical approximation

3.4.1 The WKB approximation

Consider single-well potential with classical left and right turning points $q_L(E)$ and $q_R(E)$ for given energy E: $\qquad E = V(q_L) = V(q_R)$



The WKB formula (good for small $\hbar$) reads

$$\int_{q_L}^{q_R} \mathrm{d}x\,\sqrt{2m(E-V(x))} = \hbar\pi\left(n+\tfrac{1}{2}\right), \qquad n = 0,1,2,3,\ldots$$

and provides an approximation for the quantum energy eigenvalues E_n

Remarks:

- In general a good approximation for large n.
- For the ground state energy less useful.

3.4.2 The SUSY version

Consider

$$V_\pm(x) = \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}}\,\Phi'(x)$$

and interpret last term as quantum correction to the classical potential $V_{\text{class}}(x) = \Phi^2(x)$. Apply WKB formula

$$\begin{aligned}
I &:= \sqrt{2m}\int_{q_L}^{q_R} \mathrm{d}x\,\sqrt{E - \Phi^2(x) \mp \frac{\hbar}{\sqrt{2m}}\,\Phi'(x)} \\
&\approx \sqrt{2m}\int_{x_L}^{x_R} \mathrm{d}x\,\sqrt{E-\Phi^2(x)}\left(1 \mp \frac{\hbar}{\sqrt{2m}}\,\frac{1}{2}\,\frac{\Phi'(x)}{\sqrt{E-\Phi^2(x)}}\right) \\
&\qquad \text{here} \qquad q_{L/R} \to x_{L/R} \qquad \text{where} \qquad \Phi^2(x_{L/R}) = E \\
&= \int_{x_L}^{x_R} \mathrm{d}x\,\sqrt{2m(E-\Phi^2(x))} \mp \frac{\hbar}{2}\int_{x_L}^{x_R}\mathrm{d}x\,\frac{\Phi'(x)}{\sqrt{E-\Phi^2(x)}} \\
&= \int_{x_L}^{x_R} \mathrm{d}x\,\sqrt{2m(E-\Phi^2(x))} \mp \frac{\hbar}{2}\underbrace{\int_{\Phi(x_L)}^{\Phi(x_R)}\mathrm{d}\Phi\,\frac{1}{\sqrt{E-\Phi^2}}}_{=:J}
\end{aligned}$$

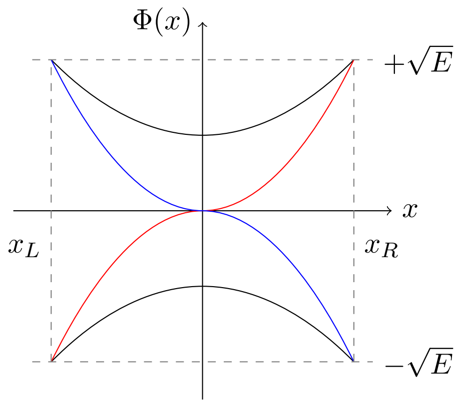
Four cases for the integral J

$$J = \int_{\Phi(x_L)}^{\Phi(x_R)}\mathrm{d}\Phi\,\frac{1}{\sqrt{E-\Phi^2}} = \arcsin\frac{\Phi(x_R)}{\sqrt{E}} - \arcsin\frac{\Phi(x_L)}{\sqrt{E}}$$

Case $\Delta = 0$: $\Phi(x_L) = \Phi(x_R) = \pm\sqrt{E} \implies J = 0$

Case $\Delta = +1$: $\Phi(x_L) = -\Phi(x_R) = -\sqrt{E} \implies J = +\pi$

Case $\Delta = -1$: $\Phi(x_L) = -\Phi(x_R) = +\sqrt{E} \implies J = -\pi$



Result:

$$I = \int_{x_L}^{x_R}\mathrm{d}x\,\sqrt{2m(E-\Phi^2(x))} \mp \frac{\hbar}{2}\,\pi\Delta$$

Hence, via WKB formula we arrive at the *Supersymmetric version of WKB* for both $V_\pm(x)$

$$\boxed{\int_{x_L}^{x_R}\mathrm{d}x\,\sqrt{2m(E-\Phi^2(x))} = \hbar\pi\left(n + \frac{1}{2} \pm \frac{\Delta}{2}\right)}$$

Remarks:

- $\Delta = +1$: $E_0^- = 0$ is exact! $E_{n+1}^- = E_n^+ > 0$ spectral symmetry conserved!
- $\Delta = -1$: $E_0^+ = 0$ is exact! $E_{n+1}^+ = E_n^- > 0$ spectral symmetry conserved!
- $\Delta = 0$: $E_n^+ = E_n^- > 0$ spectral symmetry conserved!
- For all shape invariant potentials ALL $E_n^\pm$ are exact for all Δ!
- For other systems supersymmetric version usually provides better approximations!

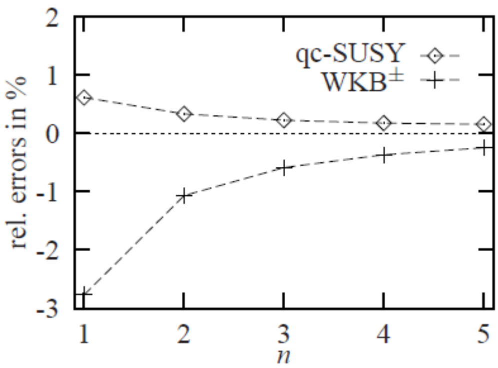


Figure 6.12: Relative errors for the good-SUSY potential $\Phi(x) = \sqrt{\hbar^2/2m}\,\sinh x$. Here the WKB approximation respects the exact relation $E_n^- = E_{n+1}^+$.

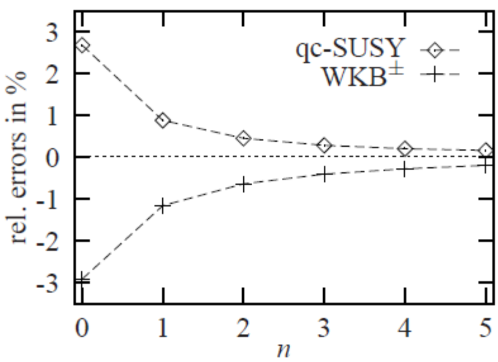


Figure 6.13: Relative errors for the broken-SUSY potential $\Phi(x) = \sqrt{\hbar^2/2m}\,\cosh x$.

Summary of section 3

1-dim. Witten model fully characterised by SUSY potential Φ
Partner potentials

$$V_{\pm}(x) = \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}}\,\Phi'(x)$$

Witten index given by

$$\Delta = \frac{1}{2}\,(\Phi_+ - \Phi_-)$$

Shape invariance

$$V_+(a_0, x) = V_-(a_1, x) + R(a_1)$$

provides exact solutions
Quasi classical SUSY approximation for spectrum of $H_\pm$

$$\int_{x_L}^{x_R} \mathrm{d}x\,\sqrt{2m(E - \Phi^2(x))} = \hbar\pi\left(n + \frac{1}{2} \pm \frac{\Delta}{2}\right)$$

SUSY transformations for continuous states (scattering) $\Longrightarrow$ Tutorial Exercise 5

4 The Darboux Method (1882)

Assumption: Let us assume we have two self-adjoint operators $H_\pm$ and one linear operator A, all acting on common Hilbert space $\mathcal{H}$ obeying the condition

$$\boxed{H_+ A = A H_-} \qquad \Longrightarrow \qquad A^\dagger H_+ = H_- A^\dagger \qquad\qquad (*)$$

Let's further assume the spectral properties of H_+ are known (we assume a purely discrete spectrum for simplicity)

$$H_+|\phi_n^+\rangle = E_n|\phi_n^+\rangle\,, \qquad n = 0, 1, 2, 3, \ldots$$

Then

$$|\phi_n^-\rangle := C_n\, A^\dagger|\phi_n^+\rangle \neq 0$$

is eigenstate of H_- with same eigenvalue E_n.
Obvious as

$$H_-|\phi_n^-\rangle = C_n H_- A^\dagger|\phi_n^+\rangle \stackrel{(*)}{=} C_n A^\dagger H_+|\phi_n^+\rangle = E_n|\phi_n^-\rangle$$

Remarks:

- States ϕ_n^+ such that $A^\dagger|\phi_n^+\rangle = 0$ do not lead to a $|\phi_n^-\rangle$. Hence, eigenvalues of H_+ associated with states $\phi_n^+ \in \ker A^\dagger$ are in general not eigenvalues of H_-
- With $A|\phi_n^-\rangle \neq 0$ we obtain an eigenstate of H_+. Let $H_-|\phi_n^-\rangle = E_n|\phi_n^-\rangle$ then

$$H_+ A|\phi_n^-\rangle \stackrel{(*)}{=} A H_-|\phi_n^-\rangle = E_n A|\phi_n^-\rangle$$

- H_- may have additional eigenvalues with eigenstates $\phi_n^- \in \ker A$, i.e. $A|\phi_n^-\rangle = 0$

Conclusion: From spectral properties of H_+ on may conclude those of H_-.
$H_\pm$ are not necessarily Schrödinger operators $\quad\Longrightarrow\quad$ Wide fields of applications

4.1 Modelling Conditionally Exactly Solvable Potentials

Let

$$H_\pm = -\frac{\hbar^2}{2m}\,\partial_x^2 + V_\pm(x) \qquad \text{on} \qquad \mathcal{H} = L^2(\mathbb{R})$$

be two 1-dim. Schrödinger Hamiltonians.

Ansatz for A:

$$A := \sum_{k=0}^{N} f_k(x)\partial_x^k$$

with $f_k : \mathbb{R} \to \mathbb{R}$ being at least twice differentiable.

Insert into defining relation $(*)$ and compare coefficients of same power of ∂_x^k
$\Longrightarrow$ Solve for the f_k's

Obviously $f_N = const.$ for convenience we choose $f_N := \hbar/\sqrt{2m}$

4.1.1 The simplest non-trivial case $N = 1$

$$A := \frac{\hbar}{\sqrt{2m}}\,\partial_x + \Phi(x) \qquad \text{with} \qquad \Phi(x) := f_0(x)\,, \qquad f_1 := \hbar/\sqrt{2m}$$

Inserting into $(*)$ results in two coupled equations

$$V_-(x) = V_+(x) - \frac{2\hbar}{\sqrt{2m}}\,\Phi'(x)$$
$$\frac{\hbar}{\sqrt{2m}} V_-'(x) + \Phi(x)V_-(x) = -\frac{\hbar^2}{2m}\Phi''(x) + \Phi(x)V_+(x)$$

Elimination of V_- results in a non-linear Riccati equation

$$\Phi^2(x) + \frac{\hbar}{\sqrt{2m}}\Phi'(x) = V_+(x) - \varepsilon\,.$$

Here $\varepsilon \in \mathbb{R}$ is a constant of integration.

Linearisation with ansatz: $\Phi(x) =: \dfrac{\hbar}{\sqrt{2m}}\dfrac{u'(x)}{u(x)}$

$$\boxed{\left[-\frac{\hbar^2}{2m}\,\partial_x^2 + V_+(x)\right] u(x) = \varepsilon u(x)}$$

Schrödinger-type equation BUT u is NOT required to be square integrable and ε is not necessarily an eigenvalue of H_+. See Tutorial Exercise 8.

Remarks:

- $H_+ = AA^\dagger + \varepsilon$, $\qquad H_- = A^\dagger A + \varepsilon \qquad$ shifted Witten model
- New potential V_- with associated Hamiltonian H_- whose spectral properties are basically known.
$$V_-(x) = \frac{\hbar^2}{m}\left(\frac{u'(x)}{u(x)}\right)^2 - V_+(x) + 2\varepsilon$$
- Condition: $u(x) \neq 0$ for all $x \in \mathbb{R}$ $\quad\Longrightarrow\quad$ No singularities!
$$\boxed{\varepsilon \leq E_0 := \min \operatorname{spec} H_+} \qquad \text{Sturm} - \text{Liouville Theory}$$
- Consider $\ker A^\dagger$: $A^\dagger|\phi_0^+\rangle = 0 \quad\Longrightarrow\quad -\dfrac{\hbar}{\sqrt{2m}}\phi_0^{+\prime}(x) + \Phi(x)\phi_0^+(x) = 0$

  $\Longrightarrow \quad \dfrac{\hbar}{\sqrt{2m}}\dfrac{\phi_0^{+\prime}(x)}{\phi_0^+(x)} = \Phi(x) = \dfrac{\hbar}{\sqrt{2m}}\dfrac{u'(x)}{u(x)} \quad\Longrightarrow\quad \phi_0^+(x) = u(x)$ nodeless

  $\Longrightarrow \quad \varepsilon = E_0$

  From now on $\varepsilon < E_0 \quad\Longrightarrow\quad \ker A^\dagger = \emptyset$.
  Complete spectrum of H_+ belongs to spectrum of H_-. $\qquad \operatorname{spec} H_+ \subset \operatorname{spec} H_-$
- Consider $\ker A$: $A|\phi_\varepsilon^-\rangle = 0 \quad\Longrightarrow\quad \phi_\varepsilon^{-\prime}(x) = -\dfrac{u'(x)}{u(x)}\phi_\varepsilon^-(x) \quad\Longrightarrow$
$$\boxed{\phi_\varepsilon^-(x) = \frac{C}{u(x)}}$$
  Assume nodeless $u(x) \to \infty$ for $x \to \pm\infty$ such that $\phi_\varepsilon^- \in L^2(\mathbb{R}) \quad\Longrightarrow$
$$\operatorname{spec} H_- = \{\varepsilon, E_0, E_1, E_2 \ldots\} = \{\varepsilon\} \cup \operatorname{spec} H_+$$

With $|\phi_n^-\rangle = C_n A^\dagger|\phi_n^+\rangle$ follows $||\phi_n^-||^2 = |C_n|^2\langle\phi_n^+|AA^\dagger|\phi_n^+\rangle = |C_n|^2\langle\phi_n^+|H_+ - \varepsilon|\phi_n^+\rangle$

Hence $|C_n|^2 = \dfrac{1}{E_n - \varepsilon} > 0$.

Summary of results: Given: Known spectral properties $\quad H_+|\phi_n^+\rangle = E_n|\phi_n^+\rangle$

$\Longrightarrow \quad H_-|\phi_n^-\rangle = E_n|\phi_n^-\rangle \quad$ and $\quad H_-|\phi_\varepsilon^-\rangle = \varepsilon|\phi_\varepsilon^-\rangle \quad$ with $\quad \varepsilon < E_0$

with conditionally exactly solvable potential

$$\boxed{V_-(x) = \frac{\hbar^2}{2m}\left(\frac{u'(x)}{u(x)}\right)^2 - V_+(x) + 2\varepsilon}$$

as $\varepsilon < E_0$ and $u(x)$ nodeless where

$$-\frac{\hbar^2}{2m}\,u''(x) + V_+(x)u(x) = \varepsilon u(x)$$

and spectral properties

$$\begin{aligned}
\operatorname{spec} H_- &= \{\varepsilon, E_0, E_1, E_2, \ldots\}\\
\phi_\varepsilon^-(x) &= \frac{C}{u(x)} \in L^2(\mathbb{R})\\
\phi_n^-(x) &= \frac{1}{\sqrt{E_n-\varepsilon}}\left(-\frac{\hbar}{\sqrt{2m}}\phi_n^{+\prime}(x) + \frac{\hbar}{\sqrt{2m}}\frac{u'(x)}{u(x)}\phi_n^+(x)\right)\\
&= \frac{\hbar}{\sqrt{2m(E_n-\varepsilon)}}\left(\frac{u'(x)}{u(x)}\phi_n^+(x) - \phi_n^{+\prime}(x)\right)
\end{aligned}$$

4.2 A family of SUSY partners of the linear harmonic oscillator

For simplicity we set $\hbar = m = \omega = 1$.

$V_+(x) = \frac{1}{2}\,x^2 \qquad$ with $\qquad E_n = (n + \frac{1}{2})$

Obviously $\varepsilon < \frac{1}{2}$

General solution of Schrödinger-like eq.
(see, e.g., Galindo & Pascual, QMI Springer 1989, p. 143 and appendix A)

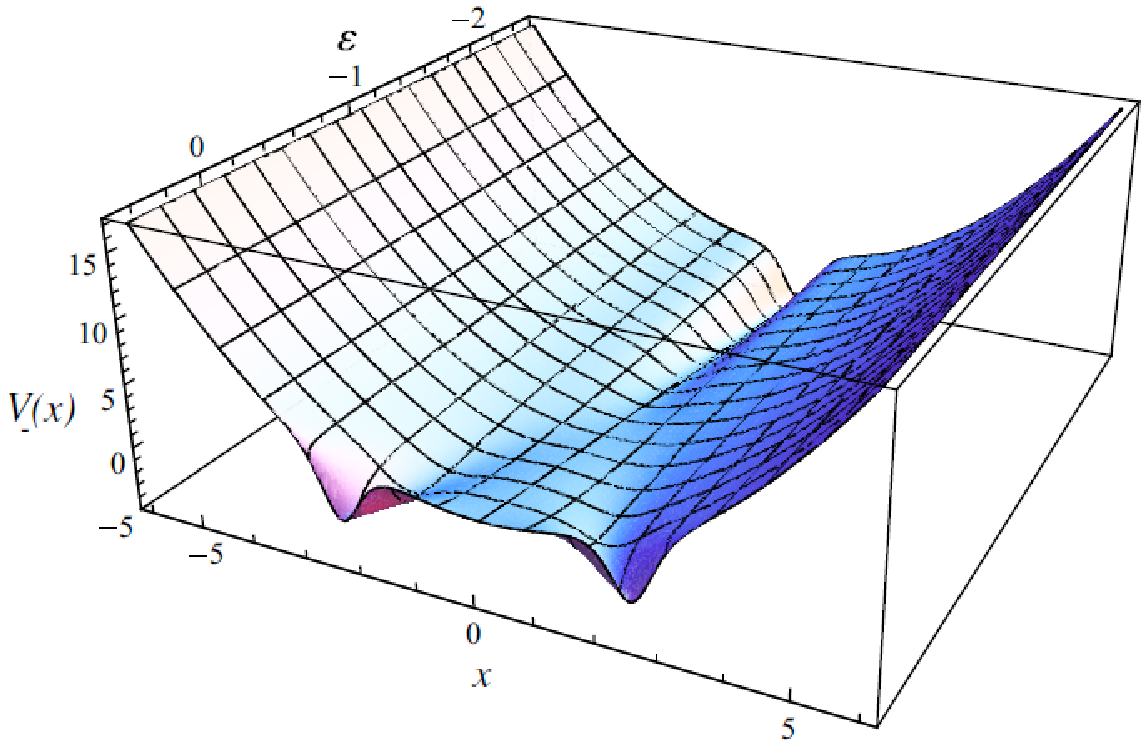
$$u(x) = \mathrm{e}^{-x^2/2}\left[\alpha\,{}_1F_1\left(\frac{1-2\varepsilon}{4},\frac{1}{2},x^2\right) + \beta\,x\,{}_1F_1\left(\frac{3-2\varepsilon}{4},\frac{3}{2},x^2\right)\right]$$

Confluent hypergeom. function:

$${}_1F_1(a,c,z) \equiv M(a,c,z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!} \qquad \text{with} \qquad (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = a(a{+}1)(a{+}2)\cdots(a{+}n{-}1)$$

For $a = -m$, $m \in \mathbb{N}_0$, this is a polynomial in z of degree m

Symmetric case $\beta = 0$



Remarks:

- Without loss of generality $\alpha = 1$
- $u(x) > 0$ for all $x \in \mathbb{R}$ $\quad\Longrightarrow\quad$ $|\beta| < \beta_c(\varepsilon) := 2\frac{\Gamma(3/4-\varepsilon/2)}{\Gamma(1/4-\varepsilon/2)}$
- $\beta = 0$: $V_-(x) = V_-(-x)$ sym. see figure above
- $\beta \in \mathbb{C}\setminus(]-\infty,-\beta_c] \cup [\beta_c,\infty[)$ allowed $\Longrightarrow$ complex potential with real spectrum
  Area of intensive research in last 20 years

Spectral properties:

$$H_+ : \quad \operatorname{spec} H_+ = \{E_0, E_1, E_2, \ldots\}, \qquad E_n = n + \tfrac{1}{2}$$

$$\phi_n^+(x) = \left(\frac{1}{\sqrt{\pi}2^n n!}\right)^{1/2} \mathrm{e}^{-x^2/2} H_n(x) \qquad \text{Hermite polynomials}$$

$$V_+(x) = \frac{1}{2}\,x^2$$

$$H_- : \quad \operatorname{spec} H_- = \{\varepsilon, E_0, E_1, E_2, \ldots\}, \qquad \varepsilon < \tfrac{1}{2} \qquad \text{arbitrary}$$

$$\phi_\varepsilon^-(x) = \frac{C\,\mathrm{e}^{x^2/2}}{{}_1F_1\left(\frac{1-2\varepsilon}{4}, \frac{1}{2}; x^2\right) + \beta x\,{}_1F_1\left(\frac{3-2\varepsilon}{4}, \frac{3}{2}; x^2\right)}$$

$$\phi_n^-(x) = \frac{\mathrm{e}^{-x^2/2}}{[\sqrt{\pi}\,2^{n+1} n!(n+1/2-\varepsilon)]^{1/2}} \left[H_{n+1}(x) + \left(\frac{u'(x)}{u(x)} - x\right) H_n(x)\right]$$

$$V_-(x) = \left[\left(\frac{u'(x)}{u(x)}\right)^2 - \frac{1}{2}x^2 + 2\varepsilon\right].$$

Special cases:

- $\varepsilon = -\frac{1}{2}$, $\beta = 0$:

  $$u(x) = \mathrm{e}^{-x^2/2}\,{}_1F_1(\tfrac{1}{2}, \tfrac{1}{2}, x^2) = \mathrm{e}^{x^2/2}\,, \qquad \tfrac{u'(x)}{u(x)} = x\,,\ \phi_n^-(x) = \phi_{n+1}^+(x)$$

- $\varepsilon = -\frac{1}{2} - 2k,\ k \in \mathbb{N}_0,\ \beta = 0$:

  $u(x) = \mathrm{e}^{-x^2/2}\,{}_1F_1(k+\frac{1}{2}, \frac{1}{2}, x^2) = \mathrm{e}_1^{x^2/2}F_1(-k, \frac{1}{2}, -x^2)$ (Hermite polynomial)
  Note: ${}_1F_1(a, c, z) = \mathrm{e}^z\,{}_1F_1(c-a, c, -z))$

  $$u(x) = \mathrm{e}^{x^2/2}\underbrace{(-1)^k\frac{k!}{(2k)!}}_{=:1/\alpha}H_{2k}(\mathrm{i}x) = \mathrm{e}^{x^2/2}H_{2k}(\mathrm{i}x)$$

  - $k = 0$: $H_0(\mathrm{i}x) = 1$ previous case
  - $k = 1$: $H_1(\mathrm{i}x) = 4(\mathrm{i}x)^2 - 2 = -4x^2 - 2$ $\Longrightarrow$ Homework
  - k arbitrary:
    $u'(x) = x\mathrm{e}^{x^2/2}H_{2k}(\mathrm{i}x) + \mathrm{i}\mathrm{e}^{x^2/2}H'_{2k}(\mathrm{i}x)\,,\quad H'_{2k}(z) = 2zH_{2k}(z) - H_{2k+1}(z)$ $\Longrightarrow$
    $\frac{u'(x)}{u(x)} = x + \mathrm{i}\frac{H'_{2k}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)} = x + \mathrm{i}2\mathrm{i}x - \mathrm{i}\frac{H_{2k+1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)} = -x - \mathrm{i}\frac{H_{2k+1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)}$

    Rational potential

    $$V_-(x) = \frac{x^2}{2} + 2\mathrm{i}x\frac{H_{2k+1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)} - \left(\frac{H_{2k+1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)}\right)^2 - 4k - 1$$

    generates spectrum $\operatorname{spec} H_- = \{-\frac{1}{2} - 2k, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$

For a complete discussion for shape-invariant potentials see GJ & P. Roy, Ann. Phys. 270 (1998) 155
Homework: Find all SUSY partners of the free particle.

Summary of section 4

- Darboux method closely related to SUSY QM but can be extended beyond
- Designing of quantum potentials with known spectral properties. More recently discussion of complex potentials (PT-symmetry)
- The family of harmonic oscillator SUSY partners also inspired new ladder operators obeying a non-linear algebra (see Exercise 9)

5 Classical Fields in $(1+1)$ Dimensions

Consider a scalar field:

$$\phi : \begin{array}{l} \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (x,t) \mapsto \phi(x,t) \end{array}$$

with vanishing variations at infinity, that is,

$$\phi' := \partial_x \phi \to 0 \qquad \text{and} \qquad \dot{\phi} := \partial_t \phi \to 0 \qquad \text{for} \qquad x, t \to \pm\infty.$$

The corresponding Lagrange density is defined as

$$\mathcal{L}(\partial\phi, \phi) := \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - U(\phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - U(\phi)$$

with a real-valued field potential U bounded from below, i.e. $U \geq 0$.
The Euler-Lagrange equation

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

then results in the classical eq. of motion

$$\partial_\mu \partial^\mu \phi + U'(\phi) = 0$$

or more explicitly

$$\boxed{\ddot{\phi} - \phi'' = -\frac{\partial U}{\partial \phi}}.$$

Examples:

- Klein-Gordon: $U(\phi) = \frac{1}{2}\phi^2$

  $\Longrightarrow \quad \partial_\mu \partial^\mu \phi + \phi = 0$
  KG equation for rel. scalar field with unit mass

- Sine-Gordon: $U(\phi) = 1 + \cos\phi$

  $\Longrightarrow \quad \ddot{\phi} - \phi'' + \sin\phi = 0$
  Instantons / Solitons

- ϕ^4-theory: $U(\phi) = \frac{1}{2}(1-\phi^2)^2$

  $\Longrightarrow \quad \ddot{\phi} - \phi'' + 2(1-\phi^2)\phi = 0$
  Phase transitions / Higgs mechanism

Conserved energy functional:

$$E[\phi] := \int_{\mathbb{R}} \mathrm{d}x \, \left[\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\phi'^2 + U(\phi)\right],$$

Homework: Show $\frac{\mathrm{d}}{\mathrm{d}t} E[\phi] = 0$

Finite energy configurations:
Now in addition we assume that $U(\phi) \to 0$ as $x \to \pm\infty$ (vacuum configuration)
That is, we assume

$$\phi_\pm := \lim_{x\to\pm\infty} \phi(x,t) \qquad \text{with} \qquad U(\phi_\pm) = 0$$

We further assume translation invariance:

$$\phi(x,t) = \phi_{\text{st}}(x - vt) \qquad\qquad \text{st} = \text{static}$$

These localised solutions are called *solitary waves*

Eq. of motion for a static solution $\phi_{\text{st}}(x)$

$$\begin{aligned} &\phi''_{\text{st}}(x) = U'(\phi_{\text{st}}(x)) \\ \Longrightarrow \qquad &\phi'_{\text{st}}(x)\phi''_{\text{st}}(x) = U'(\phi_{\text{st}}(x))\phi'_{\text{st}}(x) \\ \Longrightarrow \qquad &\tfrac{1}{2}\left[\phi'_{\text{st}}\right]^2 = U(\phi_{\text{st}}) + \varepsilon \end{aligned}$$

Recall $\phi'_{\text{st}} \to 0$ and $U(\phi_{\text{st}}) \to 0$ for $x \to \pm\infty \qquad \Longrightarrow \qquad \varepsilon = 0$

Result:

$$\boxed{\frac{1}{2}{\phi'_{\text{st}}}^2(x) = U(\phi_{\text{st}}(x))}$$

5.1 Stability of static solutions

Consider fluctuations around a static solution

$$\phi(x) = \phi_{\text{st}}(x) + \psi(x)$$

with small fluctuation ψ such that $\psi(x) \to 0 \quad \text{as} \quad x \to \pm\infty$.
That is

$$E[\phi] \approx E[\phi_{\text{st}}] + \delta E[\psi]$$

where (see tutorial)

$$\delta E[\psi] := \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x\, \psi(x)\left[-\partial_x^2 + U''(\phi_s(x))\right]\psi(x)$$

Fluctuation operator:

$$H := -\partial_x^2 + U''(\phi_{\text{st}}(x))$$

Schrödinger-like operator acting on $L^2(\mathbb{R})$.

Assume that we know the eigenmodes, that is,

$$H\psi_n = \mu_n\psi_n\,,$$

then

$$\psi(x) = \sum_n a_n\,\psi_n(x) \qquad \text{with} \qquad a_n := \int_{\mathbb{R}} \mathrm{d}x\,\psi_n^*(x)\psi(x)$$

Hence

$$\delta E[\psi] = \frac{1}{2}\sum_n \mu_n\,|a_n|^2$$

Stability:

$$\delta E[\psi] \geq 0 \qquad \Longleftrightarrow \qquad \mu_n \geq 0$$

Lemma: The "lowest" mode $n = 0$ for a stable static solution belongs to the eigenvalue $\mu_0 = 0$. This "zero"mode is given by $\psi_0(x) = C\,\phi'_{\rm st}(x)$.

Proof: We know $\frac{1}{2}{\phi'_{\rm st}}^2(x) = U(\phi_{\rm st}(x))$

$$\partial_x \qquad \Longrightarrow \qquad \phi''_{\rm st}(x) = U'(\phi_{\rm st}(x))$$

$$\partial_x \qquad \Longrightarrow \qquad \phi'''_{\rm st}(x) = U''(\phi_{\rm st}(x))\phi'_{\rm st}(x)$$

Now

$$H\psi_0(x) = C\left[-\partial_x^2 + U''(\phi_{\rm st})\right]\phi'_{\rm st} = C\left(-\phi'''_{\rm st} + U''(\phi_{\rm st})\phi'_{\rm st}\right) = 0$$

Remark: The zero mode is related to the translation invariance

$$\phi_{\rm st}(x+\delta x) = \phi_{\rm st}(x) + \phi'_{\rm st}(x)\delta x = \phi_{\rm st}(x) + \frac{\delta x}{C}\psi_0(x)$$

Fluctuation along zero mode is in essence a translation, here

$$\delta E[\phi_{\rm st}(x+\delta x) - \phi_{\rm st}(x)] = 0 \qquad \text{as} \qquad \mu_0 = 0.$$

5.2 SUSY construction of field models

Recall

$$H = -\partial_x^2 + U''(\phi_{\rm st}(x)) \geq 0$$

with vanishing lowest eigenvalue $\mu_0 = 0$. This allows to interpret

$$H \equiv H_- = -\partial_x^2 + W^2(x) - W'(x)$$

being a Witten partner Hamiltonian with SUSY potential W in units $2m = 1 = \hbar$.
Here choose W such that SUSY is unbroken.

Idea:

- Choose a SUSY potential W, e.g. one of the shape-invariant ones
- Zero mode is given by

$$\psi_0(x) = \mathcal{N}\exp\left\{-\int \mathrm{d}x\, W(x)\right\}$$

- Obtain static solution via integration

$$\phi_{\rm st}(x) = \frac{1}{C}\int \mathrm{d}x\,\psi_0(x)$$

- Use relation

$$U(\phi_{\rm st}(x)) = \frac{1}{2}{\phi'_{\rm st}}^2(x)$$

  to obtain an expression $U = U(\phi)$ by eliminating the x via previous relation $\phi_{\rm st} = \phi_{\rm st}(x)$. Choose parameter $\mathcal{N}/C$ most suitable. Finally analytically continue beyond $\phi_\pm$ to $\phi \in \mathbb{R}$.

- A field potential (theory) is found which has a stable static solution. In case of a shape-invariant W we in addition know all the fluctuation modes and their eigenvalues explicitly.

Example: $W(x) = \tanh x$ SUSY partner of free particle, has 1 bound state $\mu_0 = 0$

$$\psi_0(x) = \mathcal{N}\frac{1}{\cosh x} \qquad \text{with} \qquad \mathcal{N}/C = 2$$

$$\phi_{\rm st}(x) = 2\int \mathrm{d}x\,\frac{1}{\cosh x} = 2\arcsin(\tanh x) \qquad \Longrightarrow \qquad \sin\frac{\phi_{\rm st}}{2} = \tanh x$$

$$\phi_{\rm st}(x) \to \phi_\pm = \pm\pi \qquad \text{for} \qquad x \to \pm\infty$$

$$U(\phi_{\rm st}) = \tfrac{1}{2}{\phi'_{\rm st}}^2(x) = \tfrac{2}{\cosh^2 x} = 2(1-\tanh^2 x)$$

$$= 2(1-\sin^2\tfrac{\phi_{\rm st}}{2}) = 1 + (1 - 2\sin^2\tfrac{\phi_{\rm st}}{2}) = 1 + \cos\phi_{\rm st}$$

analytical continuation leads to

$$\text{Sine} - \text{Gordon} \qquad U(\phi) = 1 + \cos\phi$$

Tutorial: $W(x) = 2\tanh x \qquad \Longrightarrow \qquad \phi_{\rm st}(x) = \tanh x \qquad \Longrightarrow \qquad U(\phi) = \frac{1}{2}(1-\phi^2)^2$

Homework: $W(x) = \operatorname{sgn} x \qquad \Longrightarrow \qquad U(\phi) = \frac{1}{2}(1-|\phi|)^2$

Remarks:

- $W(x) = 3\tanh x \qquad \Longrightarrow$ no closed form for U, implicit relations are

$$U(\phi_{\rm st}) = \frac{2}{\cosh^6 x} = U(-\phi_{\rm st})\,, \qquad \phi_{\rm st}(x) = \frac{\tanh x}{\cosh x} + \arcsin(\tanh x)\,, \qquad \phi_\pm = \pm\frac{\pi}{2}$$

- $W(x) = 4\tanh x \qquad \Longrightarrow$ new model

$$U(\phi) = \frac{1}{2}\left[1 + 2\cos\left(\tfrac{2}{3}\arccos(\tfrac{3}{2}\phi)\right) + \tfrac{8\pi}{4}\right]^4\,, \qquad \phi_\pm = \pm\frac{2}{3}$$

- For a complete discussion on shape-inv. SUSY potentials see GJ and P. Roy, Ann. Phys. 256(1997)302. Includes also discussion on unstable fields potentials

6 Supersymmetry in Stochastic Processes

Literature on stochastic processes

1 N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, (North-Holland, 1992)

2 C.W. Gardiner, *Handbook of Stochastic Methods*, (Springer-Verlag, 1990)

6.1 The Langevin Equation

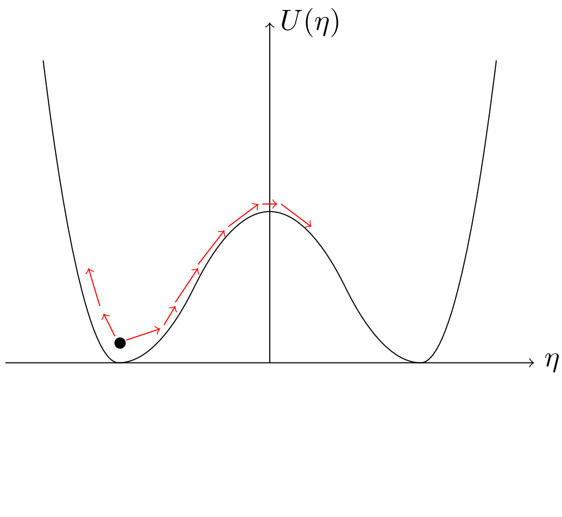
$$\boxed{\dot{\eta} = -U'(\eta) + \xi(t)}$$

Stochastic differential equation where

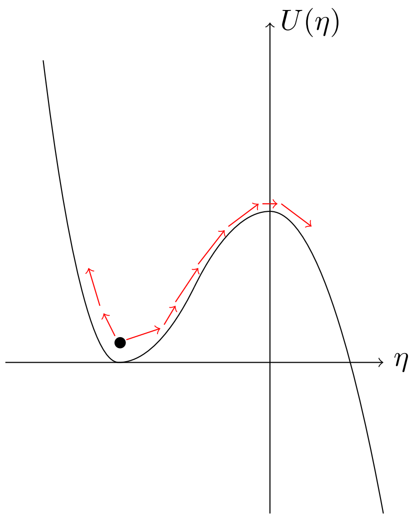
η: macroscopic degree of freedom.
For example, position of a highly overdamped motion of a Brownian particle $(\gamma\dot{\eta} \gg m\ddot{\eta})$

U: External deterministic force F or drift
$F(\eta) = -U'(\eta)$

ξ: Stochastic force (noise). For example, simulating a coupling to heat bath



(Bi-)stable System



Meta-stable System

Gaussian white noise:

$$\langle \xi(t) \rangle = 0 \qquad \langle \xi(t)\xi(t') \rangle = D\delta(t-t')$$

zero mean — No correlation in time

Diffusion constant D. For ideal heat bath $D = 2k_B T$

Idealisation of more realistic colored noise

$$\langle \xi(t)\xi(t') \rangle = \frac{D}{2\tau_c} \exp\{-|t-t'|/\tau_c\},$$

with correlation time $\tau_c > 0$. Limit $\tau_c \to 0 =$ white noise. From now on only white noise.

Average via "path integral":

$$\langle \cdot \rangle := \int_{x(0)=x_0} \mathcal{D}\xi \exp\left\{-\frac{1}{2D}\int_0^\infty \mathrm{d}\tau\, \xi^2(\tau)\right\}(\cdot)$$

In general no interest in a particular solution of the Langevin equation, but on average behaviour.

6.2 The Fokker-Planck Equation

Transition probability density:

$$m_t(x, x_0) := \langle \delta(\eta(t) - x) \rangle \qquad \text{where} \qquad x_0 := \eta(0).$$

Is the probability density to arrive at position x at time $t > 0$ for a Brownian particle starting as x_0 at time 0.

Fokker-Planck Equation:

$$\boxed{\frac{\partial}{\partial t} m_t(x, x_0) = \frac{D}{2} \frac{\partial^2}{\partial x^2} m_t(x, x_0) - \frac{\partial}{\partial x} U'(x) m_t(x, x_0)} \qquad (FP)$$

with initial condition $\qquad m_0(x, x_0) = \delta(x - x_0).$

The stationary distribution:
Assume the below limit exists, then

$$P_{\text{st}}(x) := \lim_{t \to \infty} m_t(x, x_0) \qquad \text{with} \qquad \int_{-\infty}^{+\infty} \mathrm{d}x \, P_{\text{st}}(x) = 1\,.$$

Insert in (FP):

$$0 = \frac{D}{2} \frac{\partial^2}{\partial x^2} P_{\text{st}}(x) - \frac{\partial}{\partial x} U'(x) P_{\text{st}}(x)$$

Integration:

$$\frac{D}{2} \frac{\partial}{\partial x} P_{\text{st}}(x) - U'(x) P_{\text{st}}(x) = const.$$

As $P_{\text{st}}(x)$ is normalisable we can assume $P_{\text{st}}(x) \to 0$ and $P'_{\text{st}}(x) \to 0$ as $x \to \pm\infty$.
So constant of integration should be $const. = 0$

Integration:

$$\boxed{P_{\text{st}}(x) = C \exp\left\{ -\frac{2}{D} U(x) \right\} = \mathrm{e}^{-U(x)/k_B T}}$$

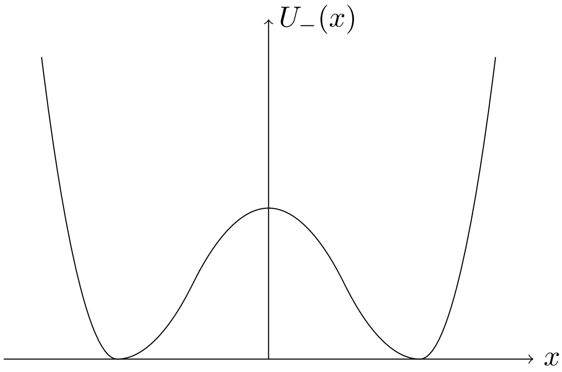
The assumption that this is normalisable implies restriction on the shape of the drift potential. Typical shapes are

Stable	Meta Stable	Unstable
$P_{\text{st}}(x)$ exists	$\lim_{t \to \infty} m_t(x, x_0) = 0$	$\lim_{t \to \infty} m_t(x, x_0) = 0$

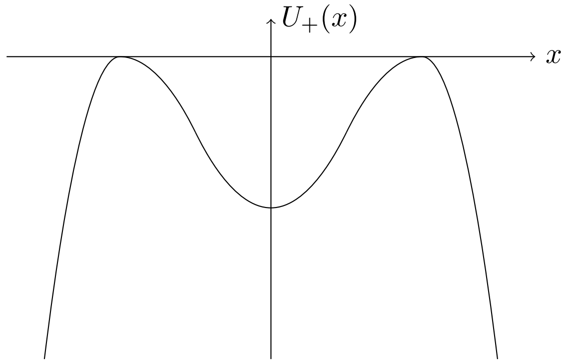
6.3 Supersymmetry of the FP equation

Consider pair of drift potentials $U_\pm$ defined via forces $U'_\pm = F_\pm := \mp\Phi(x)$ or

$$U_\pm(x) := \mp \int_0^x \mathrm{d}z\, \Phi(z) = -U_\mp(x).$$



stable



meta stable

FP equation:

$$\partial_t\, m_t^\pm(x,x_0) = \frac{D}{2}\,\partial_x^2\, m_t^\pm(x,x_0) \mp \partial_x\, \Phi(x) m_t^\pm(x,x_0) \qquad \text{with} \qquad m_0^\pm(x,x_0) = \delta(x-x_0)$$

Ansatz:

$$m_t^\pm(x,x_0) =: \exp\left\{-\frac{1}{D}\left[U_\pm(x) - U_\pm(x_0)\right]\right\} K_t^\pm(x,x_0) \qquad \text{with} \qquad K_0^\pm(x,x_0) = \delta(x-x_0)$$

leads to

$$\begin{aligned}
\partial_x\, m_t^\pm(x,x_0) &= \mathrm{e}^{-[\cdots]/D}\left(-\frac{1}{D}U'_\pm(x)K_t^\pm(x,x_0) + \partial_x K_t^\pm(x,x_0)\right) \\
&= \mathrm{e}^{-[\cdots]/D}\left(\partial_x K_t^\pm(x,x_0) \pm \frac{1}{D}\Phi(x)K_t^\pm(x,x_0)\right) \\
\partial_x^2\, m_t^\pm(x,x_0) &= \mathrm{e}^{-[\cdots]/D}\left(\partial_x^2 K_t^\pm(\cdot) \pm \frac{2}{D}\Phi(x)K_t^\pm(\cdot) + \frac{\Phi^2(x)}{D^2}K_t^\pm(\cdot) \pm \frac{\Phi'(x)}{2}K_t^\pm(\cdot)\right)
\end{aligned}$$

In FP equation multiplied by D

$$-D\partial_t K_t^\pm(x,x_0) = \left(-\frac{D^2}{2}\partial_x^2 + \frac{1}{2}\Phi^2(x) \pm \frac{D}{2}\Phi'(x)\right) K_t^\pm(x,x_0)$$

Time-dependent imaginary-time Schrödinger eq. for pair of Hamiltonians

$$H_\pm^{FP} := -\frac{D^2}{2}\partial_x^2 + \frac{1}{2}\Phi^2(x) \pm \frac{D}{2}\Phi'(x)$$

One-to-one correspondence with partner Hamiltonians of Witten model

Witten Model	$\Longleftrightarrow$	Pair of FP
$H_\pm \geq 0$		$H_\pm^{FP} \geq 0$
$\mathrm{i}t$		t
$\hbar$		D
m		1
Φ		$\frac{1}{\sqrt{2}}\Phi$

Solution: Is given by the Euclidean time evolution operator (density matrix)

$$K_t^{\pm}(x, x_0) = \langle x| \mathrm{e}^{-tH_{\pm}^{FP}/D} |x_0\rangle$$

Assume: Purely discrete spectrum for simplicity, that is,

$$H_{\pm}^{FP}|\phi_n^{\pm}\rangle = \lambda_n^{\pm}|\phi_n^{\pm}\rangle\,, \qquad n \in \mathbb{N}_0,$$

Then

$$m_t^{\pm}(x, x_0) = \exp\left\{-\frac{1}{D}\left[U_{\pm}(x) - U_{\pm}(x_0)\right]\right\} \sum_{n=0}^{\infty} \exp\left\{-\frac{1}{D}\, t\lambda_n^{\pm}\right\} \phi_n^{\pm}(x)\phi_n^{\pm *}(x_0).$$

Remarks:

- $\lambda_n^{\pm} \geq 0$ are the decay rates for $U_{\pm}$
- $\phi_n^{\pm}(x)$ are the corresponding decay modes

Stationary distribution: $\Longleftrightarrow \lambda_0^{\pm} = 0 \Longleftrightarrow$ unbroken SUSY with $\Delta = \mp 1$

$$P_{\mathrm{st}}^{\pm}(x) = \lim_{t\to\infty} m_t^{\pm}(x, x_0) = \exp\left\{-\frac{1}{D}\left[U_{\pm}(x) - U_{\pm}(x_0)\right]\right\} \phi_0^{\pm}(x)\phi_0^{\pm *}(x_0)$$

Recall

$$\phi_0^{\pm}(x) = \mathcal{N} \exp\left\{\pm\frac{\sqrt{2m}}{\hbar}\int \mathrm{d}x\, \Phi(x)\right\} = \mathcal{N} \exp\left\{\pm\frac{1}{D}\int \mathrm{d}x\, \Phi(x)\right\} = \mathcal{N} \exp\left\{-\frac{1}{D}U_{\pm}(x)\right\}$$

Hence

$$P_{\mathrm{st}}^{\pm}(x) = |\phi_0^{\pm}(x)|^2$$

Is normalisable in case of unbroken SUSY, i.e. $U_{\pm}(x) \to \infty$ fast enough.
Note, in the case of unbroken SUSY only one of below cases exist

$\Delta = +1$: $P_{\mathrm{st}}^{-}(x) = |\phi_0^{-}(x)|^2$ exists, U_- is stable, U_+ is unstable

$\Delta = -1$: $P_{\mathrm{st}}^{+}(x) = |\phi_0^{+}(x)|^2$ exists, U_+ is stable, U_- is unstable

Obviously "$P_{\mathrm{st}}^{-}(x) = \dfrac{1}{P_{\mathrm{st}}^{+}(x)}$"

Factorisation:
Recall $A = \frac{\hbar}{\sqrt{2m}}\partial_x + \Phi(x) \Longrightarrow A = \frac{1}{\sqrt{2}}(D\partial_x + \Phi(x))$, $A^\dagger = \frac{1}{\sqrt{2}}(-D\partial_x + \Phi(x))$
$\Longrightarrow \qquad H_+^{FP} = AA^\dagger \geq 0 \qquad H_-^{FP} = A^\dagger A \geq 0$

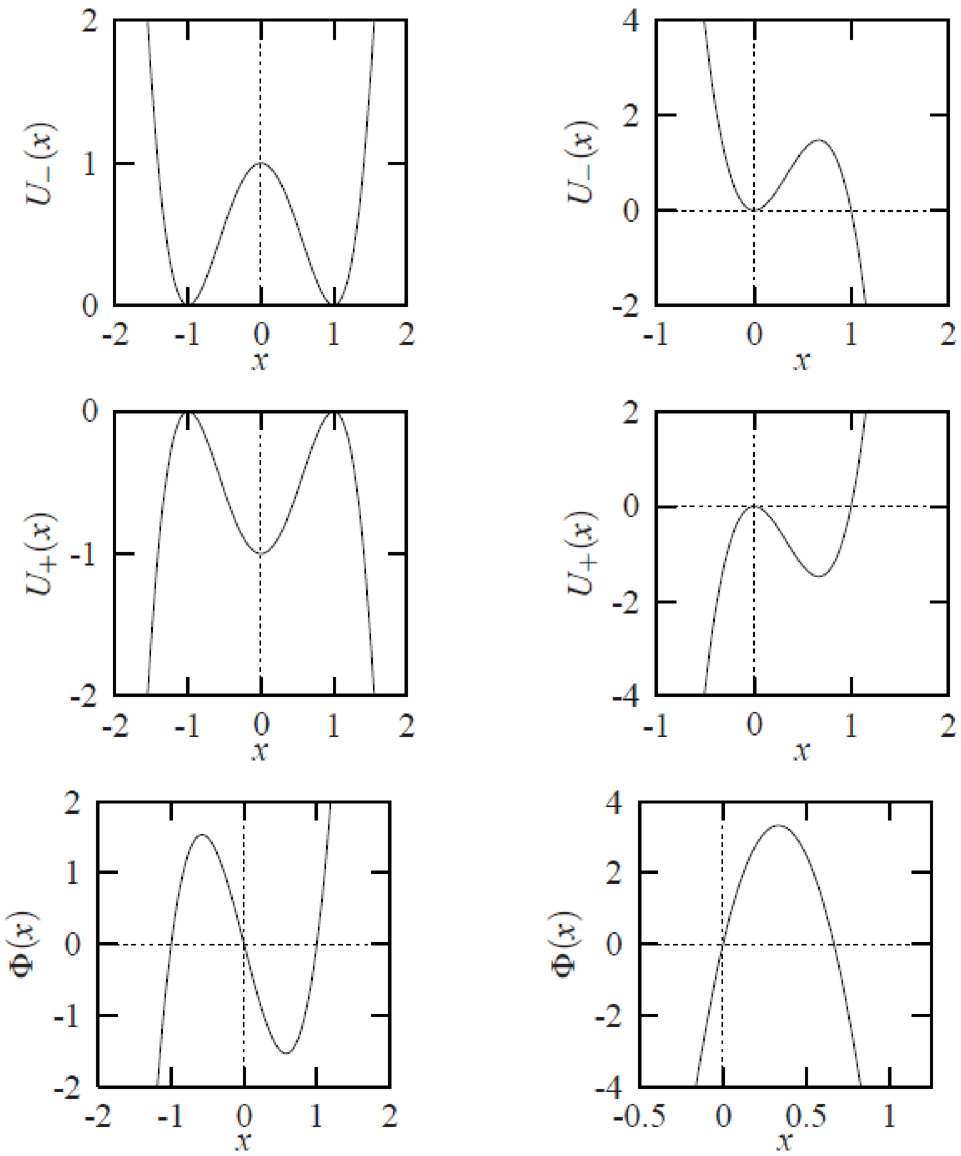


Figure 7.1: Typical drift potentials U_-, inverted drift potentials $U_+ = -U_-$ and drift coefficients $\Phi = \mp U'_\pm$ for good SUSY (*left row*) and broken SUSY (*right row*).

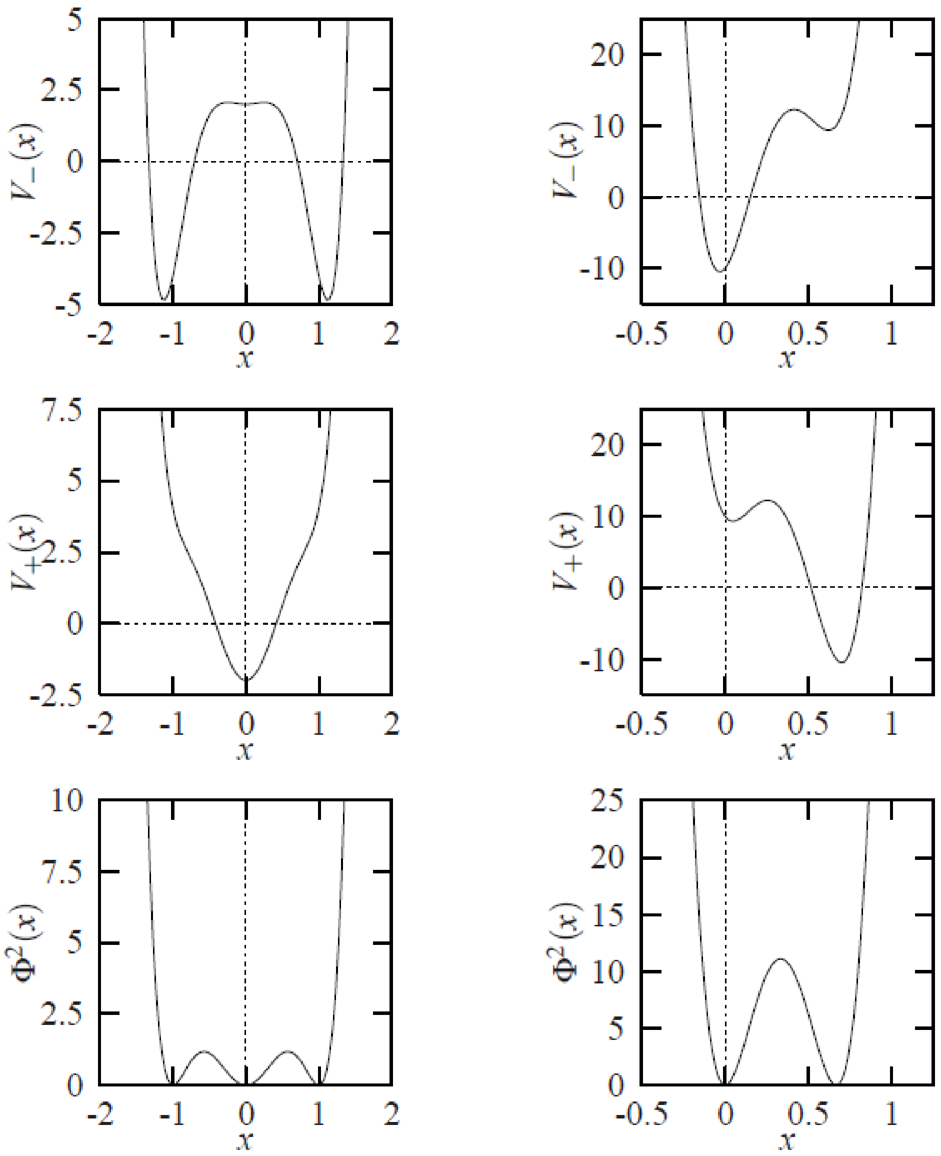


Figure 7.2: The partner potentials $V_\pm$ and Φ^2 for the drift potentials shown in Figure 7.1. Again the left row corresponds to the good-SUSY and the right to the broken-SUSY case. The diffusion constant D has been set to unity.

6.3.1 Implications of unbroken SUSY

We use convention $\Delta = +1$, hence U_- is the stable potential and U_+ is unstable.

- **Stationary distribution**: $\lambda_0^- = 0$

$$P_{\rm st}(x) = |\mathcal{N}|^2 \mathrm{e}^{-\frac{2}{D}\,U_-(x)} = |\phi_0^-(x)|^2$$

- **Decay rates:**

$$\lambda_n := \lambda_n^- = \lambda_{n-1}^+ > 0\,, \qquad n = 1, 2, 3, \dots$$

  Note: U_+ and $U_- = -U_+$ have identical decay rates!

- **SUSY transformations:** Relation between decay modes

$$\phi_{n-1}^+(x) = \frac{1}{\sqrt{2\lambda_n}}\left(D\,\frac{\partial}{\partial x} + \Phi(x)\right)\phi_n^-(x),$$
$$\phi_n^-(x) = \frac{1}{\sqrt{2\lambda_n}}\left(-D\,\frac{\partial}{\partial x} + \Phi(x)\right)\phi_{n-1}^+(x),$$

- **Transition probability density:** Spectral representation

$$\boxed{\begin{aligned} m_t^-(x,x_0) = {} & |\phi_0^-(x)|^2 + \frac{\phi_0^-(x)}{\phi_0^-(x_0)}\sum_{n=1}^{\infty}\mathrm{e}^{-\lambda_n t/D}\phi_n^-(x)\phi_n^{-*}(x_0), \\ m_t^+(x,x_0) = {} & \frac{\phi_0^-(x_0)}{\phi_0^-(x)}\sum_{n=1}^{\infty}\mathrm{e}^{-\lambda_n t/D}\phi_{n-1}^+(x)\phi_{n-1}^{+*}(x_0). \end{aligned}}$$

$\tau := D/\lambda_1$: time scale for decay of U_+ = time scale of U_- to reach $P_{\rm st}$.

6.3.2 Implications of broken SUSY

- **Decay rates:**

$$\lambda_n := \lambda_n^- = \lambda_n^+ > 0\,, \qquad n = 0, 1, 2, 3, \dots$$

  As before: U_+ and $U_- = -U_+$ have identical decay rates! No stationary distribution.

- **SUSY transformations:**

$$\phi_n^\pm(x) = \frac{1}{\sqrt{2\lambda_n}}\left(\pm D\,\frac{\partial}{\partial x} + \Phi(x)\right)\phi_n^\mp(x).$$

- **Transition probability density:** Spectral representation

$$\boxed{m_t^\pm(x,x_0) = \exp\left\{\pm\frac{1}{D}\left[U_-(x) - U_-(x_0)\right]\right\}\sum_{n=0}^{\infty}\mathrm{e}^{-\lambda_n t/D}\phi_n^\pm(x)\phi_n^{\pm*}(x_0),}$$

  Note: $\exp\left\{\pm\frac{1}{D}\left[U_-(x) - U_-(x_0)\right]\right\} = \exp\left\{-\frac{1}{D}\left[U_\pm(x) - U_\pm(x_0)\right]\right\}$

6.3.3 Some examples

$$\left.\begin{aligned} \Phi_1(x) &= a\,\mathrm{sgn}\,x \\ \Phi_2(x) &= a\tanh x \\ \Phi_3(x) &= a - \mathrm{e}^{-x} \end{aligned}\right\} \qquad \text{for} \qquad a > 0 \qquad \text{unbroken SUSY (Homework)}$$

Case 3:

Zero mode: $\phi_0^-(x) = \mathcal{N}\exp\left\{-\int \mathrm{d}x\,\Phi_3(x)\right\} = \mathcal{N}\exp\left\{-ax - \mathrm{e}^{-x}\right\}$

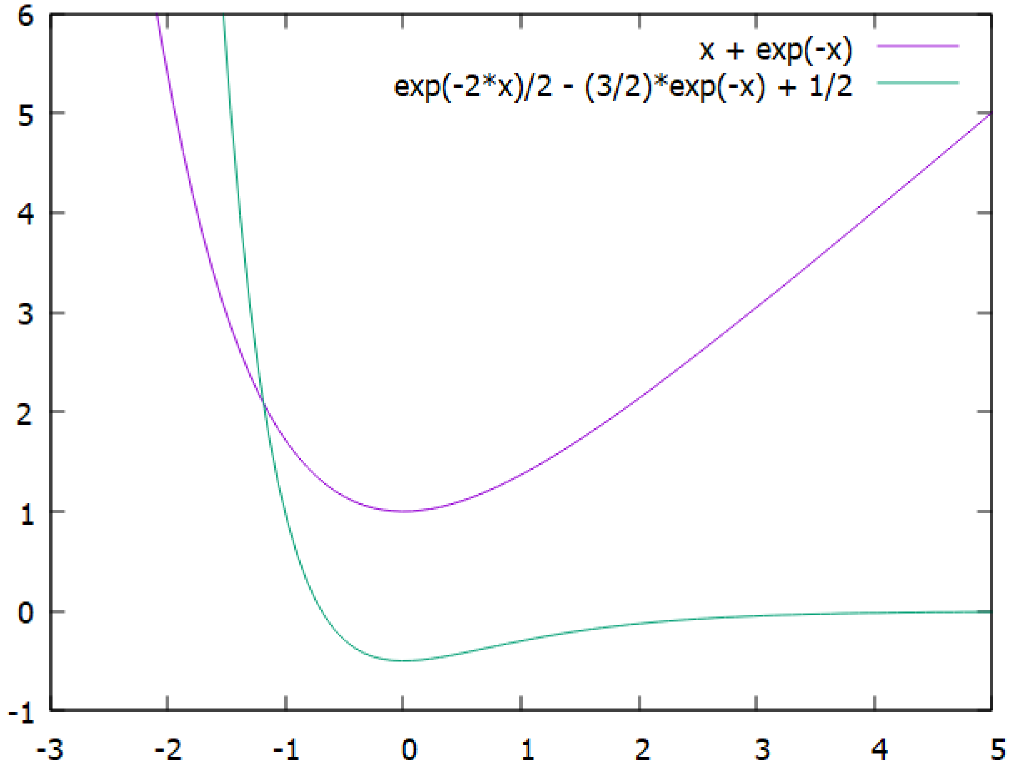
Stationary distribution: $P_{\mathrm{st}}(x) = |\phi_0^-(x)|^2 = \mathcal{N}^2\exp\left\{-2ax - 2\mathrm{e}^{-x}\right\}$

Drift potential: $U_-(x) = \int \mathrm{d}x\,\Phi_3(x) = ax + \mathrm{e}^{-x}$

Partner potentials: $V_\pm(a,x) = \frac{1}{2}\Phi_3^2(x) \pm \frac{1}{2}\Phi_3'(x) = \frac{1}{2}\mathrm{e}^{-2x} - (a \mp \frac{1}{2})\mathrm{e}^{-x} + \frac{1}{2}a^2$

Note: $V_+(a,x) = V_-(a-1,x) + a - \frac{1}{2}$ (shape-inv. Morse potential)

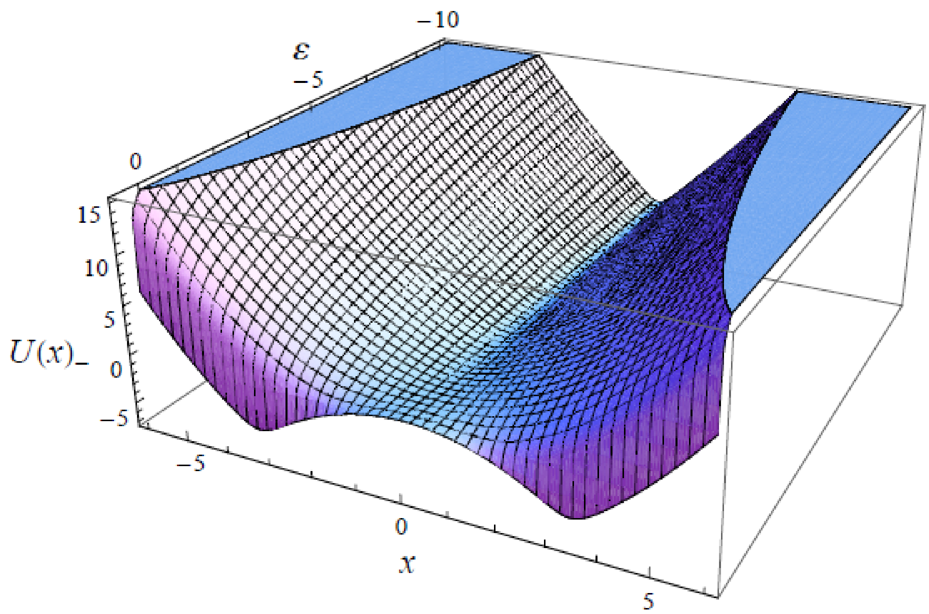
Obviously $\lambda_1 = a - \frac{1}{2}$ if $a > 1$ or $\lambda_1 = \frac{a^2}{2}$ if $0 < a < 1$ (V_- has only 1 bound state).



Additional Homework: Discuss $\Phi(x) = x$

As for the Witten model one can construct conditionally exactly solvable drift potentials (see Book)

Family of stable drift potentials related to the harmonic oscillator



Summary of Section 6

- SUSY naturally appears in Fokker-Planck equation.
- Also for the Langevin equation (see the book section 7.3)
- Diffusion in drift potential U_- and in its inverted potential $U_+ = -U_-$ are closely related.
- For broken SUSY both have same decay rates.
- For unboken SUSY (U_- stable) equilibrium distribution is given by the SUSY ground state, relaxation times into equilibrium are also the decay rates for U_+.
- "Supersymmetric theory of stochastic dynamics" first introduced (1979-1982) by G. Parisi (Nobel price 2021) and N. Sourlas.

7 Supersymmetry in the Pauli-Hamiltonian

7.1 $N = 1$ SUSY of Pauli-Hamiltonian in 3 Dimensions

Spin $\frac{1}{2}$ particle with mass $m > 0$ and charge e ($e < 0$ for electron) in external el.-magn. field characterised by

Vector potential: $\vec{A}(\vec{r}, t)$
Scalar potential: $\phi(\vec{r}, t)$
Hilbert space: $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$

Phenomenological Pauli-Hamiltonian

$$H := \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}(\vec{r}, t)\right)^2 - \vec{\mu}_S \cdot \vec{B}(\vec{r}, t) + e\phi(\vec{r}, t)$$

Magnetic field: $\vec{B}(\vec{r}, t) = \vec{\nabla} \times \vec{A}(\vec{r}, t)$
Electric field: $\vec{E}(\vec{r}, t) = \vec{\nabla}\phi(\vec{r}, t)$
Gauge transformations:

$$\tilde{\phi}(\vec{r}, t) = \phi(\vec{r}, t) - \frac{1}{c}\dot{\chi}(\vec{r}, t)\,, \quad \tilde{\vec{A}}(\vec{r}, t) = \vec{A}(\vec{r}, t) + \vec{\nabla}\chi(\vec{r}, t)\,, \quad \tilde{\psi}(\vec{r}, t) = \mathrm{e}^{\frac{\mathrm{i}e}{\hbar c}\chi(\vec{r}, t)}\psi(\vec{r}, t)$$

Spin: $\vec{S} := \frac{\hbar}{2}\vec{\sigma}$ with Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \vec{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}.$$

Magnetic moment: $\vec{\mu}_S := g\frac{e}{2mc}\vec{S} = g\frac{e\hbar}{4mc}\vec{\sigma} = \frac{g}{2}\,\mathrm{sgn}\,e\,\mu_B\,\vec{\sigma}$
Bohr magenton: $\mu_B := \frac{|e|\hbar}{2mc}$ g: Landé g-factor interaction term $H_S := -\vec{\mu}_S \cdot \vec{B}$
For electrons $e < 0$:

- non-relativistic SUSY: $g = 2$
- relativistic Dirac SUSY theory: $g = 2$
- standard model theory: $g = 2.002\,319\,304\,363\,22(46)$
- experiment: $g = 2.002\,319\,304\,363\,56(35)$

We know from Tutorial 1: From now on $\phi = 0$ and $\dot{\vec{A}} = 0$

- $N = 1$ SUSY with $Q = \frac{1}{\sqrt{4m}}\left(\vec{P} - \frac{e}{c}\vec{A}\right) \cdot \vec{\sigma} = Q^\dagger$
- No Witten operator but helicity operator $\Lambda = \frac{m\vec{V} \cdot \vec{\sigma}}{\sqrt{2mH}} = \mathrm{sgn}\,Q$
- Velocity operator $\vec{V} = \dot{\vec{r}} = \frac{\mathrm{i}}{\hbar}[H, \vec{r}] = \frac{1}{m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)$
- SUSY Pauli-Hamiltonian:

$$\boxed{H = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{2mc}\vec{\sigma} \cdot \vec{B}}$$

Homework:
Show $\Lambda^\dagger = \Lambda$, $\Lambda^2 = 1$, $[\Lambda, H] = 0$, $Q = \mathrm{sgn}\,Q\,|Q| = \sqrt{\frac{H}{2}}\,\Lambda$

7.2 $N = 2$ SUSY of Pauli-Hamiltonian in 2 Dimensions

Vector potential: $\vec{A}(x_1, x_2) = \begin{pmatrix} a_1(x_1, x_2) \\ a_2(x_1, x_2) \end{pmatrix}$

Magnetic field: $\vec{B}(x_1, x_2) = B(x_1, x_2)\vec{e}_3\,,\qquad B(x_1, x_2) = \partial_1 a_2(x_1, x_2) - \partial_2 a_1(x_1, x_2)$

Hilbert space: $\mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2 = \mathcal{H}^+ \oplus \mathcal{H}^-$

Witten operator: $W = \sigma_3$, $\qquad \mathcal{H}^\pm = L^2(\mathbb{R}^2)$ spin up/down subspace

Supercharge:

$$Q := A \otimes \sigma^+ = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with

$$A := \frac{1}{\sqrt{2m}} \left[\left(P_1 - \frac{e}{c} a_1 \right) \mp \mathrm{i} \left(P_2 - \frac{e}{c} a_2 \right) \right]$$

$\Longrightarrow \qquad N = 2$ SUSY as $Q \neq Q^\dagger$

Hamiltonian: $H := \{Q, Q^\dagger\} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix}$

Calculation:

$$\begin{aligned} AA^\dagger &= \tfrac{1}{2m} \left[\left(P_1 - \tfrac{e}{c} a_1 \right) \mp \mathrm{i} \left(P_2 - \tfrac{e}{c} a_2 \right) \right] \left[\left(P_1 - \tfrac{e}{c} a_1 \right) \pm \mathrm{i} \left(P_2 - \tfrac{e}{c} a_2 \right) \right] \\ &= \tfrac{1}{2m} \left[\left(P_1 - \tfrac{e}{c} a_1 \right)^2 + \left(P_2 - \tfrac{e}{c} a_2 \right)^2 \mp \mathrm{i} \left[\left(P_2 - \tfrac{e}{c} a_2 \right) \left(P_1 - \tfrac{e}{c} a_1 \right) - \left(P_1 - \tfrac{e}{c} a_1 \right) \left(P_2 - \tfrac{e}{c} a_2 \right) \right] \right] \\ &= \tfrac{1}{2m} \left(\vec{P} - \tfrac{e}{c} \vec{A} \right)^2 \mp \mathrm{i} \tfrac{e}{2mc} \left([P_1, a_2] + [a_1, P_2] \right) \\ &= \tfrac{1}{2m} \left(\vec{P} - \tfrac{e}{c} \vec{A} \right)^2 \mp \mathrm{i} \tfrac{e\hbar}{2mc\mathrm{i}} \left(\partial_1 a_2 - \partial_2 a_1 \right) \\ &= \tfrac{1}{2m} \left(\vec{P} - \tfrac{e}{c} \vec{A} \right)^2 \mp \tfrac{e\hbar}{2mc} B(x_1, x_2) \end{aligned}$$

Similarly $\qquad A^\dagger A = \tfrac{1}{2m} \left(\vec{P} - \tfrac{e}{c} \vec{A} \right)^2 \pm \tfrac{e\hbar}{2mc} B(x_1, x_2)$

Result:

$$H = \frac{1}{2m} \left(\vec{P} - \frac{e}{c} \vec{A} \right)^2 \mp \frac{e\hbar}{2mc} B(x_1, x_2) \sigma_3$$

$\Longrightarrow \qquad N = 2$ SUSY of Pauli-Hamiltonian with $g = \pm 2$.

Witten parity eigenstates are eignestates of S_3.

From now on we consider only upper sign $g = +2$ and electrons $e = -|e|$.

$$H = \frac{1}{2m} \left(\vec{P} - \frac{e}{c} \vec{A} \right)^2 + \mu_B B(x_1, x_2) \sigma_3$$

Magnetic flux:

$$F := \int_{\mathbb{R}^2} \mathrm{d}x_1 \mathrm{d}x_2 \, B(x_1, x_2)$$

and assume $|F| < \infty$, that is, B is bounded with compact support

Aharonov-Casher theorem: (see Tutorial 14)

- Ground state energy: $E_0 = 0 \qquad \Longrightarrow \qquad$ SUSY unbroken
- Degeneracy of E_0: $\qquad d = \left[\!\left[\frac{|F|}{\Phi_0} \right]\!\right]$
  Here $[\![z]\!] := \max\limits_{n \in \mathbb{N}_0} \{n | n < z\}$, largest integer stricly less then z.
  And $\Phi_0 := 2\pi \frac{\hbar c}{|e|}$ represents the flux quantum.
- All d ground states belong either
  to $\qquad \mathcal{H}^-$ for $F > 0$, spin-down states
  or $\qquad \mathcal{H}^+$ for $F < 0$, spin-up states
- SUSY implies that all states with $E > 0$ are pairwise ($\uparrow\downarrow$) degenerate due to existing SUSY transformations. Unpaired spins can only exist on the ground state level.
- Witten index:
  $$\Delta = \dim \ker A^\dagger A - \dim \ker AA^\dagger = d \, \mathrm{sgn}\, F \approx \frac{F}{\Phi_0}$$
  Topological invariant as details of B are irrelevant and only total flux through $\mathbb{R}^2$ is essential!

7.3 Paramagnetism of non-interacting electrons in 2D

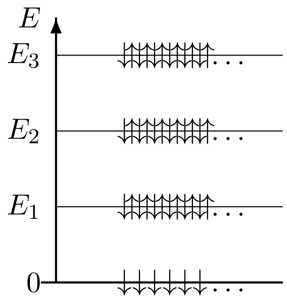
Consider a 2-dim. gas of N non-interacting electrons at $T = 0$

- **Ground state:** Is characterised by integrated density of states

$$\Theta(\varepsilon_F - H)$$

  With 1-particle Hamiltonian $\quad H = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 + \mu_B B(x_1, x_2)\sigma_3$
  and Fermi energy ε_F defined via $\quad \mathrm{Tr}\,\Theta(\varepsilon_F - H) = N$.
  All states up to ε_F are occupied.

- **Typical N-particle ground state:**
  Assumption that ε_F is between two Landau levels (case $B = const.$)
  All levels either fully occupied or empty



- **Magnetisation:** Recall magnetic moment of single electron $\vec{\mu}_S = -\mu_B \vec{\sigma}$

$$\begin{aligned} M &:= \mu_B\left(N_\downarrow - N_\uparrow\right) \qquad N_{\uparrow\downarrow} : \text{No. of occupied } \uparrow\downarrow \text{ states} \\ &= -\mu_B \mathrm{Tr}\left[\sigma_3 \Theta(\varepsilon_F - H)\right] \\ &= \mu_B \tilde{\Delta}(\varepsilon_F) \qquad \text{IDOS regulated Witten index} \\ &= \mu_B \Delta \qquad \text{under above assumption} \\ &= \mu_B\, d\,\mathrm{sgn}\, F \approx \mu_B \tfrac{F}{\Phi_0} \qquad \text{topological invariant} \end{aligned}$$

- **Simplifying assumptions:** $B(x_1, x_2) = B > 0$ constant magn. field on finite area $\mathcal{A}$
  $\mathcal{A} \subset \mathbb{R}^2$ with $\mathcal{A} := \{(x_1, x_2) \in \mathbb{R}^2 | -\ell/2 \leq x_i \leq \ell/2, i = 1, 2\} \implies F = B\ell^2 > 0$
  magnetisation: $\quad M = \mu_B \dfrac{B\ell^2}{\Phi_0}$
  specific magnetisation: $\dfrac{M}{\ell^2} = \mu_B \dfrac{B}{\Phi_0} = \mu_B \dfrac{|e|B}{2\pi\hbar c}$

- **Paramagnetic Susceptibility:** of the 2-dim. electron gas

$$\boxed{\chi := \frac{1}{\ell^2}\frac{\partial M}{\partial B} = \mu_B \frac{|e|}{2\pi\hbar c} = \frac{e^2}{4\pi m c^2}}$$

Remarks:

- Result independent of electron density (ε_f) and magnetic field strength (B)!

- Derivation uses full single-particle Pauli-Hamiltonian

$$H \equiv H^{(2)} = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 + \mu_B B \sigma_3$$

  Standard textbook use the free Hamiltonian with spin term

$$H_0 := \frac{1}{2m}\vec{P}^2 + \mu_B B \sigma_3$$

  but arrive at same result!!! $\quad \implies \quad$ "Topological Invariance"

7.4 Paramagnetism of non-interacting electrons in 3D

Homogeneous magnetic field: $\vec{B} = B\vec{e}_3$ on $\mathcal{A}$ as before
Macroscopic Volume: $V = \ell^2 \ell_3$ $\quad$ ℓ_3 is extension in x_3-direction
Single particle Hamiltonian:

$$H^{(3)} = \frac{P_3^2}{2m} + H^{(2)}$$

Free motion in x_3-direction but eigenvalues of P_3 are quantised as $\ell_3 < \infty$
periodic boundary conditions allow only certain wavelengths

$$p_3 = \hbar k_3 \qquad \text{with} \qquad k_3 = \frac{2\pi}{\ell_3}\, n\,, \qquad n \in \mathbb{Z}$$

For the non-interaction electron gas all k_3 are occupied where

$$|k_3| < k_F := \frac{\sqrt{2m\varepsilon_f}}{\hbar} \qquad \text{Fermi wave number}$$

Number of occupied k_3: $\quad 2n_{\max} = \frac{k_F \ell_3}{\pi}\,, \qquad -n_{\max} < n < n_{\max}$
Each eigenvalue k_3 contributes to magnetisation the 2-dim. result

$$M^{(2)} = \mu_B \frac{B\ell^2}{\Phi_0}$$

Total magnetisation:

$$M^{(3)} = 2n_{\max} M^{(2)} = \frac{k_F \ell_3}{\pi} \mu_B \frac{B\ell^2}{\Phi_0}$$

Specific magnetisation:

$$\frac{M^{(3)}}{V} = \frac{k_F B}{\pi} \frac{\mu_B}{\Phi_0} = \frac{e^2}{4\pi^2 mc^2}\, k_F\, B$$

Paramagnetic Susceptibility: Is dimensionless!

$$\chi^{(3)} = \frac{1}{V} \frac{\partial M^{(3)}}{\partial B} = \frac{e^2}{4\pi^2 mc^2}\, k_F = \left(\frac{\alpha}{2\pi}\right)^2 a_0\, k_F$$

Bohr radius: $a_0 := \frac{\hbar^2}{me^2}$
Fine structure constant: $\alpha := \frac{e^2}{\hbar c}$

7.5 The textbook approach

Calculate spectral density of a free particle in a box: $V = L^3$ $\quad$ using $\quad H_0 = \frac{\vec{P}^2}{2m}$

- Eigenfunctions:

$$\psi(\vec{r}) = \left(\tfrac{1}{2\pi}\right)^{3/2} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}}\,, \qquad k_j = \tfrac{2\pi}{L}\, n_j\,, \qquad n_j \in \mathbb{Z}\,, \qquad j = 1, 2, 3$$

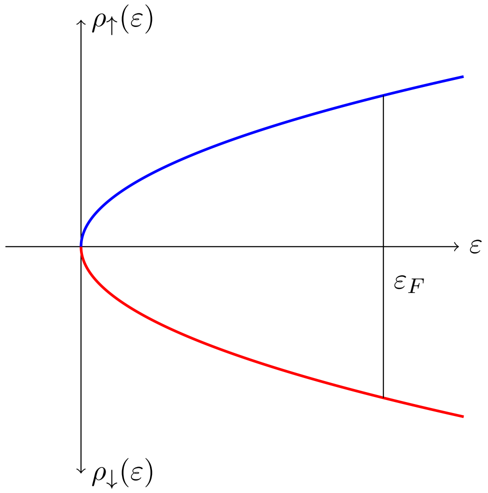
- Volume taken by one state in k-space: $\Omega_0 := \left(\frac{2\pi}{L}\right)^3$
- Volume of sphere in k-space: $\mathrm{d}\Omega = 4\pi k^2 \mathrm{d}k$
  with $\varepsilon(\vec{k}) = \frac{\hbar^2 \vec{k}^2}{2m} \quad \Rightarrow \quad k = |\vec{k}| = \sqrt{\frac{2m\varepsilon}{\hbar^2}} \quad \Rightarrow \quad \mathrm{d}\varepsilon = \frac{\hbar^2 k}{m} \mathrm{d}k$
  Hence $\quad \mathrm{d}\Omega = 4\pi \frac{2m\varepsilon}{\hbar^2} \frac{m}{\hbar^2 k} \mathrm{d}\varepsilon = 4\pi \frac{m}{\hbar^2} \sqrt{\frac{2m\varepsilon}{\hbar^2}} \mathrm{d}\varepsilon$
- Spectral density: number of states in the sphere

$$D(\varepsilon) := \frac{1}{\Omega_0} \frac{\mathrm{d}\Omega}{\mathrm{d}\varepsilon} = \frac{V}{8\pi^3} 4\pi \frac{m}{\hbar^2} \sqrt{\frac{2m\varepsilon}{\hbar^2}} = \frac{Vm}{2\pi^2 \hbar^2} \sqrt{\frac{2m\varepsilon}{\hbar^2}}$$

- Specific spectral density:

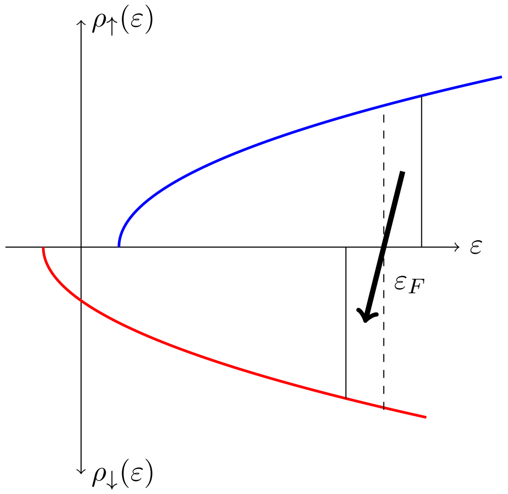
$$\rho(\varepsilon) := \frac{D(\varepsilon)}{V} = \frac{m}{2\pi^2\hbar^2}\sqrt{\frac{2m\varepsilon}{\hbar^2}} = \frac{m}{2\pi^2\hbar^2}\,k$$

Graphical representation for spin-up/-down electrons



Switch on magnetic field: Using free Hamiltonian with spin term only

$$H_0 \qquad \Rightarrow \qquad H_0 = \frac{\vec{P}^2}{2m} + \mu_B B\sigma_3$$



$N_\uparrow \quad \to \quad N_\uparrow - \mu_B B\rho(\varepsilon_F)V$
$N_\downarrow \quad \to \quad N_\downarrow - \mu_B B\rho(\varepsilon_F)V$
Magnetisation: $M^{(3)} = \mu_B(N_\downarrow - N_\uparrow) = 2\mu_B^2\rho(\varepsilon_F)VB$
Susceptibility: $\chi^{(3)} = 2\mu_B^2\rho(\varepsilon_F) = \frac{e^2\hbar^2}{4m^2c^2}\frac{m}{\pi^2\hbar^2}\,k_F = \frac{e^2}{4\pi^2mc^2}\,k_F$

Result is identical to the SUSY derivation.
Surprisingly the wrong use of the free Hamiltonian with spin term is sufficient.
The spectral free density actually changes drastically to Landau levels.
Nevertheless the net magnetisation is NOT sensitive to such approximation.

Recall $M = \mu_B\,\Delta$ is related to the Witten index, i.e. a topological invariant.

8 Supersymmetry in the Dirac-Hamiltonian

8.1 The Dirac equation

Problem: (see e.g. F. Schwabl, "QM für Fortgeschrittene")
Schrödinger eq. allows for a probabilistic interpretation but is no *relativistic* description. Klein-Gordon eq. $E^2 = \vec{p}^2c^2 + m^2c^4$ is covariant and relativistic, but does not allow for a probabilistic interpretation.

Dirac's ansatz:

$$H := c\vec{\alpha}\cdot\vec{p} + \beta mc^2$$

with $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T$ and β being arbitrary, not necessarily, numbers.

Quantisation: $E \to H$ and $\vec{p} \to -\mathrm{i}\hbar\vec{\nabla}$ results in

$$\begin{aligned} H^2 &= -c^2\hbar^2\alpha_k\alpha_l\partial_k\partial_l - \mathrm{i}\hbar mc^2(\alpha_k\beta + \beta\alpha_k)\partial_k + \beta^2m^2c^4 \\ &= -\tfrac{1}{2}c^2\hbar^2(\alpha_k\alpha_l + \alpha_l\alpha_k)\partial_k\partial_l - \mathrm{i}\hbar mc^2(\alpha_k\beta + \beta\alpha_k)\partial_k + \beta^2m^2c^4 \end{aligned}$$

Compare with KG relation $E^2 = \vec{p}^2c^2 + m^2c^4$ led Dirac to the conclusion

$$\left.\begin{aligned} \{\alpha_k, \alpha_l\} &= 2\delta_{kl} \\ \{\alpha_k, \beta\} &= 0 \\ \beta^2 &= 1 \end{aligned}\right\} \quad \text{Dirac matrices, Dirac algebra}$$

Further properties: $H = H^\dagger \quad \Longrightarrow \quad \alpha_k = \alpha_k^\dagger \quad$ and $\quad \beta = \beta^\dagger$

Consider: $\mathrm{Tr}\,\alpha_k = \mathrm{Tr}\,\alpha_k\beta^2 = -\mathrm{Tr}\,\beta\alpha_k\beta = -\mathrm{Tr}\,\alpha_k \quad \Longrightarrow \quad \mathrm{Tr}\,\alpha_k = 0$

Similar $\mathrm{Tr}\,\beta = \mathrm{Tr}\,\beta\alpha_k^2 = -\mathrm{Tr}\,\alpha_k\beta\alpha_k = -\mathrm{Tr}\,\beta \quad \Longrightarrow \quad \mathrm{Tr}\,\beta = 0$

- Pauli representation: 4×4-matrices

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \text{or} \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Weyl representation:

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{via} \quad U_W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- Supersymmetric representation:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{via} \quad U_S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$$

- Free Dirac equation: $H = \mathrm{i}\hbar\partial_t$

$$\boxed{\mathrm{i}\hbar\partial_t\Psi(\vec{r}, t) = (c\vec{\alpha}\cdot\vec{p} + \beta mc^2)\Psi(\vec{r}, t)}$$

  Ψ: Dirac spinor, lives in $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$

$$\Psi(\vec{r}, t) = \begin{pmatrix} \psi_1(\vec{r}, t) \\ \psi_2(\vec{r}, t) \\ \psi_3(\vec{r}, t) \\ \psi_4(\vec{r}, t) \end{pmatrix}$$

- Free Dirac Hamiltonian: (Pauli representation)

$$\boxed{H_0 := c\vec{\alpha}\cdot\vec{p} + \beta mc^2 = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{pmatrix}}$$

- Free massless Dirac Hamiltonian: (Weyl representation)

$$H_0 = c\vec{\alpha}\cdot\vec{p} = \begin{pmatrix} c\vec{\sigma}\cdot\vec{p} & 0 \\ 0 & -c\vec{\sigma}\cdot\vec{p} \end{pmatrix}$$

$\Longrightarrow$ Weyl eq. $\mathrm{i}\hbar\partial_t\Psi = c\vec{\sigma}\cdot\vec{p}\Psi\,, \qquad \Psi \in L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$

- Free Dirac Hamiltonian in 1D and 2D:

$$H_0 = -\mathrm{i}\hbar c\,(\sigma_1\partial_1 + \sigma_2\partial_2) + \sigma_3 mc^2$$

- Charged Dirac particle in electromagnetic potentials:
via minimal coupling $\vec{p} \to \vec{p} - \frac{e}{c}\vec{A}$ and $\mathrm{i}\hbar\partial_t \to \mathrm{i}\hbar\partial_t - e\phi_{el}$

$$\boxed{H = c\vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) + \beta mc^2 + e\phi_{el}}$$

- Scalar potentials: $V(\vec{r}) = \beta\phi_{sc}(\vec{r})$
- Dirac oscillator:

$$H = c\vec{\alpha}\cdot(\vec{p} + \beta\mathrm{i}m\omega\vec{r}) + \beta mc^2 = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot(\vec{p} - \mathrm{i}m\omega\vec{r}) \\ c\vec{\sigma}\cdot(\vec{p} + \mathrm{i}m\omega\vec{r}) & -mc^2 \end{pmatrix}$$

More details: B. Thaller, "The Dirac Equation" (Springer, Berlin, 1992)

8.2 Supersymmetric Dirac operators

Recall: $N = 2$ SUSY with Witten operator now on $\mathcal{H} = L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

SUSY Hamiltonian:

$$H_S := \{Q, Q^\dagger\} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

Definition:
Let

$$Q_1 := Q + Q^\dagger = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} = Q_1^\dagger \qquad \text{and} \qquad \mathcal{M} := \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} = \mathcal{M}^\dagger \geq 0$$

then

$$\boxed{H_D := Q_1 + \mathcal{M}W}$$

is called *supersymmetric Dirac operator* if $[Q_1, \mathcal{M}] = 0 = [W, \mathcal{M}]$.
That is,

$$\boxed{H_D = \begin{pmatrix} M_+ & A \\ A^\dagger & -M_- \end{pmatrix} \qquad \text{with} \qquad AM_- = M_+A\,, \qquad A^\dagger M_+ = M_- A^\dagger\,.}$$

Example: $A := c\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A}) = A^\dagger\,, \qquad M_\pm = mc^2 \otimes 1$

$$H_D = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A}) \\ c\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A}) & -mc^2 \end{pmatrix} = c\vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) + \beta mc^2$$

Charged Dirac particle in magnetic field.

Properties:

- Consider

$$\begin{aligned} H_D^2 &= (Q_1 + \mathcal{M}W)^2 = \begin{pmatrix} M_+ & A \\ A^\dagger & -M_- \end{pmatrix} \begin{pmatrix} M_+ & A \\ A^\dagger & -M_- \end{pmatrix} \\ &= \begin{pmatrix} AA^\dagger + M_+^2 & M_+A - AM_- \\ A^\dagger M_+ - M_- A^\dagger & A^\dagger A + M_-^2 \end{pmatrix} = \begin{pmatrix} AA^\dagger + M_+^2 & 0 \\ 0 & A^\dagger A + M_-^2 \end{pmatrix} \end{aligned}$$

  Let $m > 0$ be an arbitrary mass-like parameter and define

$$H_+ := \frac{1}{2mc^2} AA^\dagger \,, \qquad H_- := \frac{1}{2mc^2} A^\dagger A \,,$$

  Rescale supercharges

$$Q := \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} , \qquad Q^\dagger = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}$$

  and set

$$H_{SUSY} := \frac{1}{2mc^2} \left(H_D^2 - \mathcal{M}^2 \right) = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} .$$

  Then we obtain a $N = 2$ SUSY QM system with $W = \beta$

$$H_{SUSY} = \{Q, Q^\dagger\} \,, \qquad \{Q, W\} = 0 \,, \qquad Q^2 = 0 = (Q^\dagger)^2 \,.$$

- Let $U_{FW} := a_+ + W \operatorname{sgn} Q_1 \, a_-$ be unitary transformation with $a_\pm := \sqrt{\frac{1}{2} \pm \frac{\mathcal{M}}{2|H_D|}}$
  Then (see tutorial)

$$H_D^{FW} := U_{FW} H_D U_{FW}^\dagger = \begin{pmatrix} \sqrt{AA^\dagger + M_+^2} & 0 \\ 0 & -\sqrt{A^\dagger A + M_-^2} \end{pmatrix} = \beta |H_D|$$

  U_{FW} diagonalises H_D and is called *Foldy–Wouthuysen transformation.*

  Hence with

$$H_D^{FW} \widetilde{\Psi_n^\pm} = E_n^\pm \widetilde{\Psi_n^\pm} \qquad \text{and} \qquad \Psi_n^\pm := U_{FW}^\dagger \widetilde{\Psi_n^\pm} \qquad \Longrightarrow \qquad H_D \Psi_n^\pm = E_n^\pm \Psi_n^\pm$$

  The subspaces $\mathcal{H}^\pm$ are the eigenspaces of H_D for positive and negative energies, respectively.

  Observation: In many cases $M_\pm = mc^2$ and $A = A^\dagger$, that is $H_{NR} := H_\pm = \frac{A^2}{2mc^2}$

$$H_D^{FW} = \beta mc^2 \sqrt{1 + \frac{H_{NR}}{2mc^2}}$$

  Hence H_{NR} is the non-relativistic limit of H_D in those cases as

$$H_D^{FW}\big|_{\mathcal{H}^+} = mc^2 + H_{NR} + O\left(1/mc^2\right)$$

- Spectral properties: Note $[H_+, M_+] = 0 = [H_-, M_-]$

  Let $\quad H_\pm \phi_n^\pm = \varepsilon_n \phi_n^\pm \quad$ and $\quad M_\pm \phi_n^\pm = m_n c^2 \phi_n^\pm \quad$ with $\quad \varepsilon_n, m_n > 0$

  Hence we have

$$\boxed{E_n^\pm = \pm\sqrt{2mc^2 \varepsilon_n + m_n c^2} \,, \qquad \widetilde{\Psi_n^+} = \begin{pmatrix} \phi_n^+ \\ 0 \end{pmatrix} , \qquad \widetilde{\Psi_n^-} = \begin{pmatrix} 0 \\ \phi_n^- \end{pmatrix}}$$

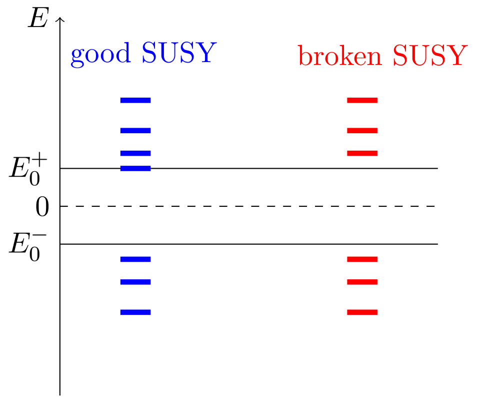
For unbroken SUSY ($\varepsilon_0 = 0$) in addition we have

$$E_0^+ = \langle\phi_0^+|M_+|\phi_0^+\rangle \quad \text{if} \quad \phi_0^+ \in \mathcal{H}^+ \text{exists with} \quad A^\dagger\phi_0^+ = 0$$

and/or

$$E_0^- = -\langle\phi_0^-|M_-|\phi_0^-\rangle \quad \text{if} \quad \phi_0^- \in \mathcal{H}^- \text{exists with} \quad A\phi_0^- = 0$$

The spectrum of a supersymmetric Dirac Hamiltonian is symmetric about zero with the exception at E_0^+ and/or E_0^- if SUSY is unbroken.



The spectral properties of H_D follow from those of the SUSY partners $H_\pm$ and $M_\pm$. In all most all case, $M_\pm = mc^2$ or $M_\pm = 0$.

Note: In general $A \sim \vec{p}$, hence $H_\pm \sim \vec{p}^2$, i.e. the relativistic problem may be reduced to a non-relativistic Pauli-like problem.

Example: Electron in magnetic field results in $H_D^{FW} = \beta mc^2\sqrt{1 + \frac{2H_P}{mc^2}}$

Dirac: $\quad H_D = c\vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) + \beta mc^2$

Pauli: $\quad H_P = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}$

- SUSY transformations for $\varepsilon_n > 0$:

Recall $\quad \phi_n^+ = \dfrac{1}{\sqrt{2mc^2\varepsilon_n}}A\phi_n^- \quad$ and $\quad \phi_n^- = \dfrac{1}{\sqrt{2mc^2\varepsilon_n}}A^\dagger\phi_n^+$.

Hence $\quad \widetilde{\Psi_n^+} = \dfrac{1}{\sqrt{\varepsilon_n}}Q\widetilde{\Psi_n^-} \quad$ and $\quad \widetilde{\Psi_n^-} = \dfrac{1}{\sqrt{\varepsilon_n}}Q^\dagger\widetilde{\Psi_n^+}$

Obvious as

$$Q\,\widetilde{\Psi_n^-} = \frac{1}{\sqrt{2mc^2}}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \phi_n^- \end{pmatrix} = \frac{1}{\sqrt{2mc^2}}\begin{pmatrix} A\phi_n^- \\ 0 \end{pmatrix} = \sqrt{\varepsilon_n}\,\widetilde{\Psi_n^+}$$

$$Q^\dagger\,\widetilde{\Psi_n^+} = \frac{1}{\sqrt{2mc^2}}\begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}\begin{pmatrix} \phi_n^+ \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2mc^2}}\begin{pmatrix} 0 \\ A^\dagger\phi_n^+ \end{pmatrix} = \sqrt{\varepsilon_n}\,\widetilde{\Psi_n^-}$$

8.3 The free Dirac Hamiltonian

Choose: $\quad A := c\vec{\sigma}\cdot\vec{p} = A^\dagger\,, \quad M_\pm := mc^2 \quad$ on $\quad \mathcal{H}^\pm = L^2(\mathbb{R}^2)\otimes\mathbb{C}^2$

$$H_D = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{pmatrix} \qquad \text{Pauli reps.}$$

With $A^\dagger A = c^2(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) = c^2\vec{p}^2 = AA^\dagger$ we have

$$H_\pm = \frac{1}{2mc^2}c^2\vec{p}^2 = \frac{\vec{p}^2}{2m} \qquad \text{free non-rel. particle on } \mathcal{H}^\pm$$

$\varepsilon_0 = 0 \in \operatorname{spec} H_\pm \qquad \Longrightarrow \qquad$ SUSY unbroken

Eigenspinors: Plane waves

$$\phi^\pm_{\vec{k}\lambda}(\vec{r}) = \left(\frac{1}{2\pi\hbar}\right)^{3/2} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}}\,\chi_\lambda(\vec{k})\,, \qquad \vec{k} \in \mathbb{R}^3\,, \qquad \lambda \in \{-1,+1\}\,,$$

with eigenvalues $\varepsilon_{\vec{k}} = \dfrac{\hbar^2\vec{k}^{\,2}}{2m}$ and 2-spinor $\chi_\lambda(\vec{k})$

Helicity eigenspinors: Let $k := |\vec{k}|$ and

$$\chi_{+1}(\vec{k}) := \frac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k_1 - \mathrm{i}k_2 \\ k - k_3 \end{pmatrix} \qquad \text{and} \qquad \chi_{+1}(k\vec{e}_3) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,,$$

$$\chi_{-1}(\vec{k}) := \frac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k_3 - k \\ k_1 + \mathrm{i}k_2 \end{pmatrix} \qquad \text{and} \qquad \chi_{-1}(k\vec{e}_3) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,,$$

Recall helicity operator $\Lambda := \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$ in eigenspace with fixed $\vec{k}$, $\Lambda_{\vec{k}} := \dfrac{\vec{\sigma}\cdot\vec{k}}{k}$

Lemma: Above spinors are ortho-normal eigenspinors of $\Lambda_{\vec{k}}$, that is,

$$\Lambda_{\vec{k}}\chi_\lambda(\vec{k}) = \lambda\chi_\lambda(\vec{k})\,, \qquad \lambda = \pm 1\,, \qquad ||\chi_\lambda||^2 = (\chi^*_\lambda)^T\chi_\lambda = 1\,, \qquad (\chi^*_{-1})^T\chi_{+1} = 0\,.$$

Proof:
Consider $\vec{\sigma}\cdot\vec{k} = \begin{pmatrix} k_3 & k_1 - \mathrm{i}k_2 \\ k_1 + \mathrm{i}k_2 & -k_3 \end{pmatrix} \qquad \Longrightarrow$

$$\begin{aligned}
\vec{\sigma}\cdot\vec{k}\chi_{+1}(\vec{k}) &= \tfrac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k_3 & k_1 - \mathrm{i}k_2 \\ k_1 + \mathrm{i}k_2 & -k_3 \end{pmatrix}\begin{pmatrix} k_1 - \mathrm{i}k_2 \\ k - k_3 \end{pmatrix} \\
&= \tfrac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k_3(k_1 - \mathrm{i}k_2) + (k_1 - \mathrm{i}k_2)(k - k_3) \\ k_1^2 + k_2^2 - k_3(k - k_3) \end{pmatrix} \\
&= \tfrac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k(k_1 - \mathrm{i}k_2) \\ k(k - k_3) \end{pmatrix} = k\chi_{+1}(\vec{k})
\end{aligned}$$

$$\begin{aligned}
\vec{\sigma}\cdot\vec{k}\chi_{-1}(\vec{k}) &= \tfrac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k_3 & k_1 - \mathrm{i}k_2 \\ k_1 + \mathrm{i}k_2 & -k_3 \end{pmatrix}\begin{pmatrix} k_3 - k \\ k_1 + \mathrm{i}k_2 \end{pmatrix} \\
&= \tfrac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k_3(k_3 - k) + k_1^2 + k_2^2 \\ (k_1 + \mathrm{i}k_2)(k_3 - k) - k_3(k_1 + \mathrm{i}k_2) \end{pmatrix} \\
&= \tfrac{1}{\sqrt{2k(k-k_3)}}\begin{pmatrix} k(k - k_3) \\ -k(k_1 + \mathrm{i}k_2) \end{pmatrix} = -k\chi_{-1}(\vec{k})
\end{aligned}$$

The ortho-normal part is homework.

Summary:

$$\begin{aligned}
&H_\pm\ \phi^\pm_{\vec{k}\lambda}(\vec{r}) = \varepsilon_{\vec{k}}\ \phi^\pm_{\vec{k}\lambda}(\vec{r}) && \text{with} && \varepsilon_{\vec{k}} = \tfrac{\hbar^2\vec{k}^{\,2}}{2m}\,, \quad \vec{k} \in \mathbb{R}^3\,, \\
&\Lambda\ \phi^\pm_{\vec{k}\lambda}(\vec{r}) = \lambda\ \phi^\pm_{\vec{k}\lambda}(\vec{r}) && \text{with} && \lambda = \pm 1\,.
\end{aligned}$$

$$\boxed{E^\pm_{\vec{k}\lambda} = \pm\sqrt{\hbar^2c^2\vec{k}^{\,2} + m^2c^4}\,, \qquad \widetilde{\Psi}^+_{\vec{k}\lambda}(\vec{r}) = \begin{pmatrix} \phi^+_{\vec{k}\lambda}(\vec{r}) \\ 0 \end{pmatrix}\,, \qquad \widetilde{\Psi}^-_{\vec{k}\lambda}(\vec{r}) = \begin{pmatrix} 0 \\ \phi^-_{\vec{k}\lambda}(\vec{r}) \end{pmatrix}}$$

Explicit form of FW transformation:
Consider subspace with fixed $\vec{k}$ and λ and set $\epsilon(k) := \sqrt{\hbar^2c^2k^2 + m^2c^4}$

- With $\quad |H_D|\widetilde{\Psi^{\pm}_{\vec{k}\lambda}} = \epsilon(k)\widetilde{\Psi^{\pm}_{\vec{k}\lambda}} \qquad \Longrightarrow \qquad a_\pm = \sqrt{\frac{1}{2} \pm \frac{mc^2}{2\epsilon(k)}}$
- $\operatorname{sgn} Q_1 = \frac{Q_1}{\sqrt{Q_1^2}}, \quad Q_1 = \begin{pmatrix} 0 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}, \quad Q_1^2 = \begin{pmatrix} c^2\vec{p}^2 & 0 \\ 0 & c^2\vec{p}^2 \end{pmatrix} = c^2\vec{p}^2 \otimes \mathbf{1}$
  $\operatorname{sgn} Q_1 = \frac{1}{\sqrt{c^2\vec{p}^2}} \begin{pmatrix} 0 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Lambda \\ \Lambda & 0 \end{pmatrix} = \Lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $W \operatorname{sgn} Q_1 = \Lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- $U_{FW} = a_+ + W \operatorname{sgn} Q_1 a_- = a_+ + a_-\Lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U_{FW}^\dagger = a_+ - a_-\Lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- The "electron" solution
  $\Psi^+_{\vec{k}\lambda}(\vec{r}) = U^\dagger_{FW}\widetilde{\Psi^+_{\vec{k}\lambda}}(\vec{r}) = \left[a_+ - \lambda a_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \begin{pmatrix} \phi^+_{\vec{k}\lambda}(\vec{r}) \\ 0 \end{pmatrix}$

$$\boxed{\Psi^+_{\vec{k}\lambda}(\vec{r}) = \begin{pmatrix} a_+\phi^+_{\vec{k}\lambda}(\vec{r}) \\ \lambda a_-\phi^+_{\vec{k}\lambda}(\vec{r}) \end{pmatrix}}$$

- The "positron" solution

$$\boxed{\Psi^-_{\vec{k}\lambda}(\vec{r}) = \begin{pmatrix} -\lambda a_-\phi^-_{\vec{k}\lambda}(\vec{r}) \\ a_+\phi^-_{\vec{k}\lambda}(\vec{r}) \end{pmatrix}}$$

- SUSY transformations: $A = c\vec{\sigma}\cdot\vec{p} = A^\dagger$, $\varepsilon_{\vec{k}} = \frac{\hbar^2k^2}{2m}$
  $A\ \phi^\pm_{\vec{k}\lambda} = c\hbar\vec{\sigma}\cdot\vec{k}\ \phi^\pm_{\vec{k}\lambda} = c\hbar k\lambda\ \phi^\mp_{\vec{k}\lambda} = \lambda\sqrt{2mc^2\varepsilon_{\vec{k}}}\ \phi^\mp_{\vec{k}\lambda}$ (λ is phase only!)
  $Q^\dagger\widetilde{\Psi^+_{\vec{k}\lambda}} = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix} \begin{pmatrix} \phi^+_{\vec{k}\lambda} \\ 0 \end{pmatrix} = \lambda\sqrt{\varepsilon_{\vec{k}}} \begin{pmatrix} 0 \\ \phi^-_{\vec{k}\lambda} \end{pmatrix} = \lambda\sqrt{\varepsilon_{\vec{k}}}\,\widetilde{\Psi^-_{\vec{k}\lambda}}$
  $Q\widetilde{\Psi^-_{\vec{k}\lambda}} = \cdots = \lambda\sqrt{\varepsilon_{\vec{k}}}\,\widetilde{\Psi^+_{\vec{k}\lambda}}$
- Free Dirac particle in SUSY representation:
  Now $\quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad$ and $\qquad \beta = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$

$$H_D = c\vec{\alpha}\cdot\vec{p} + \beta mc^2 = \begin{pmatrix} 0 & c\vec{\sigma}\cdot\vec{p} - \mathrm{i}mc^2 \\ c\vec{\sigma}\cdot\vec{p} + \mathrm{i}mc^2 & 0 \end{pmatrix}$$

  Hence

$$A := c\vec{\sigma}\cdot\vec{p} - \mathrm{i}mc^2 \qquad \text{and} \qquad M_\pm := 0 \qquad \Longrightarrow$$

$$H_+ = \frac{AA^\dagger}{2mc^2} = \frac{A^\dagger A}{2mc^2} = H_- \qquad \text{or} \qquad H_\pm = \frac{c^2\vec{p}^2}{2mc^2} + \frac{m^2c^4}{2mc^2} \geq \frac{1}{2}mc^2 > 0$$

  Here SUSY is broken with

$\varepsilon_{\vec{k}} = \frac{\hbar^2\vec{k}^2}{2m} + \frac{1}{2}mc^2$	shifted SUSY spectrum
$\phi^\pm_{\vec{k}\lambda}(\vec{r}) = \left(\frac{1}{2\pi\hbar}\right)^{3/2} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}}\ \chi_\lambda(\vec{k})$	same eigenspinors
$E^\pm_{\vec{k}} = \pm\sqrt{2mc^2\varepsilon_{\vec{k}}} = \pm\sqrt{c^2\hbar^2k^2 + m^2c^4}$	same Dirac spectrum

8.4 The Dirac oscillator

$$H = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot(\vec{p}-\mathrm{i}m\omega\vec{r}) \\ c\vec{\sigma}\cdot(\vec{p}+\mathrm{i}m\omega\vec{r}) & -mc^2 \end{pmatrix}$$

Is obviously SUSY Dirac Hamiltonian

$$A = c\vec{\sigma}\cdot(\vec{p}-\mathrm{i}m\omega\vec{r})\,,\qquad M_\pm = mc^2 \qquad \Longrightarrow \qquad AM_- = M_+A\,,\qquad A^\dagger M_+ = M_-A^\dagger$$

Homework: Show

$$\begin{aligned} AA^\dagger &= c^2\left(\vec{p}^2 + m^2\omega^2\vec{r}^2 + 3mc^3\hbar\omega + 2mc^2\omega\vec{L}\cdot\vec{\sigma}\right) = 2mc^2H_+ \\ A^\dagger A &= c^2\left(\vec{p}^2 + m^2\omega^2\vec{r}^2 - 3mc^3\hbar\omega - 2mc^2\omega\vec{L}\cdot\vec{\sigma}\right) = 2mc^2H_- \end{aligned}$$

Partner Hamitonians are SUSY Pauli Hamiltonians

$$\begin{aligned} H_\pm &= \tfrac{\vec{p}^2}{2m} + \tfrac{m}{2}\omega^2\vec{r}^2 \pm \left(\tfrac{3}{2}\hbar\omega + \hbar\omega\vec{L}\cdot\vec{\sigma}\right) \\ &= \tfrac{\vec{p}^2}{2m} + \tfrac{m}{2}\omega^2\vec{r}^2 \pm \hbar\omega\left(K+\tfrac{1}{2}\right) \end{aligned}$$

Recall spin orbit operator $K := \vec{L}\cdot\vec{\sigma}+1$

Eigenvalues of K are given by: $-\kappa = s|\kappa| = s(j+\frac{1}{2}) = \begin{cases} \ell+1 & \text{for} \quad s=+1 \\ -\ell & \text{for} \quad s=-1 \end{cases}$ or $j = \ell + \frac{s}{2}$

Eigenvalues of $H_\pm$:

$$\varepsilon^\pm_{njs} = \hbar\omega\left(2n+\ell+\tfrac{3}{2}\right) \pm \hbar\omega\left[s(j+\tfrac{1}{2})+\tfrac{1}{2}\right]$$

More explicit

$$\begin{aligned} \varepsilon^-_{njs} &= \hbar\omega\left(2n+j-\tfrac{s}{2}+\tfrac{3}{2}-sj-\tfrac{s}{2}-\tfrac{1}{2}\right) = \hbar\omega\left[2n+j+1-s(j+1)\right] \\ \varepsilon^+_{njs} &= \hbar\omega\left(2n+j-\tfrac{s}{2}+\tfrac{3}{2}+sj+\tfrac{s}{2}+\tfrac{1}{2}\right) = \hbar\omega\left[2(n+1)+j+sj\right] > 0 \end{aligned}$$

SUSY unbroken with ground state energy

$$\varepsilon^-_{0j1} = 0 \qquad \infty\text{−degenerate as} \qquad j = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots$$

Spectral relation between SUSY partners

$$\varepsilon^+_{njs} = \varepsilon^-_{n+1,j-1,-s}$$

Eigenvalues of the Dirac oscillator

$$\begin{aligned} E^-_{njs} &= -mc^2\left[1+\frac{2\hbar\omega}{mc^2}[2n+j+1-s(j+1)]\right]^{1/2} \\ E^+_{njs} &= mc^2\left[1+\frac{2\hbar\omega}{mc^2}[2(n+1)+j+sj]\right]^{1/2} \end{aligned}$$

8.5 One-dimensional Dirac Hamiltonians

- The free Dirac particle on the real line

$$H = c\sigma_1 p + \sigma_3 mc^2 = \begin{pmatrix} mc^2 & cp \\ cp & -mc^2 \end{pmatrix} \qquad \text{on} \qquad \mathcal{H} = L^2(\mathbb{R})\otimes\mathbb{C}^2$$

Obvious: $\quad A = cp = A^\dagger\,,\qquad M_\pm = mc^2\,,\qquad H_\pm = \frac{A^2}{2mc^2} = \frac{p^2}{2m} \geq 0$

- The Dirac oscillator on the real line

$$H = c\sigma_1(p + \mathrm{i}m\omega x\sigma_3) + \sigma_3 mc^2 = \begin{pmatrix} mc^2 & c(p - \mathrm{i}m\omega x) \\ c(p + \mathrm{i}m\omega x) & -mc^2 \end{pmatrix}$$

Obvious: $\quad A = c(p - \mathrm{i}m\omega x) = -\mathrm{i}\sqrt{2mc^2\hbar\omega}\, a\,, \qquad M_\pm = mc^2$

$$H_\pm = \tfrac{p^2}{2m} + \tfrac{m}{2}\omega^2 x^2 \pm \tfrac{1}{2}\hbar\omega = \hbar\omega\left(a^\dagger a + \tfrac{1}{2} \pm \tfrac{1}{2}\right)$$

Hence

$$\phi_n^+ = \langle x|n-1\rangle\,, \qquad \phi_n^- = \langle x|n\rangle\,, \qquad \varepsilon_n := \hbar\omega n\,, \qquad n = 1, 2, 3, \ldots$$

in addition $n = 0$ for H_- only.

$$E_n^\pm = \pm\sqrt{m^2c^4 + 2mc^2\varepsilon_n} = \pm mc^2\sqrt{1 + \frac{2\varepsilon_n}{mc^2}}$$

- The relativistic Witten model
  Generalisation of Dirac oscillator with $m\omega x \to \sqrt{2m}\Phi(x)$

$$H = c\sigma_1(p + \mathrm{i}\sqrt{2m}\Phi(x)\sigma_3) + \sigma_3 mc^2 = \begin{pmatrix} mc^2 & c(p - \mathrm{i}\sqrt{2m}\Phi(x)) \\ c(p + \mathrm{i}\sqrt{2m}\Phi(x)) & -mc^2 \end{pmatrix}$$

Obvious: $\quad A = c(p - \mathrm{i}\sqrt{2m}\Phi(x))\,, \qquad M_\pm = mc^2$

$$H_\pm = \frac{p^2}{2m} + \Phi(x)^2 \pm \frac{\hbar}{\sqrt{2m}}\Phi'(x)$$

Assume unbroken SUSY with $\varepsilon_0 = 0 \in \operatorname{spec} H_-$ and $\varepsilon_n > 0 \in \operatorname{spec} H_+$ then

$$E_0^- = -mc^2 \qquad \text{and} \qquad E_n^\pm = \pm mc^2\sqrt{1 + \frac{2\varepsilon_n}{mc^2}}$$

Remarks:

  - Whenever the non-relativistic Witten model can be solved, one also has a solution of the relativistic Witten model.
  - Application of the SUSY WKB formula results in an approximation for the relativistic Witten model via $E^2 = 2mc^2\varepsilon + m^2c^4$.
    Let $W(x) := \sqrt{2mc^2}\,\Phi(x)$, then $A = cp - \mathrm{i}W(x)$ and

$$\int_{x_L}^{x_R} \mathrm{d}x \sqrt{E^2 - m^2c^4 - W^2(x)} = c\hbar\pi\left(n + \frac{1}{2} \pm \frac{\Delta}{2}\right)$$

    with $W^2(x_{R/L}) = E^2 - m^2c^4$.

For a general discussion see GJ, Eur. Phys. J. Plus 135 (2020) 464 (13pp)

8.6 Relativistic Hamiltonians with arbitray spin

The Dirac Hamiltonian describes the relativistic dynamics of spin-$\frac{1}{2}$ particles.
How about particles with other spin?
Goal is to find relativistic eq. allowing for a probability interpretation, that is, being of the form

$$\mathrm{i}\hbar\partial_t\Psi = H\Psi\,, \qquad \mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2(2s+1)}\,, \qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

The general form of such a Hamiltonian is given by

$$H = \beta m + \mathcal{E} + \mathcal{O}\,, \qquad \text{with} \qquad \beta^2 = 1\,.$$

Here m denotes the mass of the particle.
$\mathcal{E}$ and $\mathcal{O}$ denote the *even* and *odd* parts of the Hamiltonian, respectively. That is,

$$[\beta, \mathcal{E}] = 0\,, \qquad \{\beta, \mathcal{O}\} = 0\,.$$

With $\mathcal{M} := m + \beta\mathcal{E}$ the general Hamiltonian then reads

$$H_s = \beta\mathcal{M} + \mathcal{O} \qquad \text{with} \qquad \begin{array}{lll} H_s = H_s^\dagger & \text{for} \quad s = \frac{1}{2}, \frac{3}{2}, \ldots\,, & \textit{Fermions} \\ H_s = \beta H_s^\dagger \beta & \text{for} \quad s = 0, 1, 2, \ldots\,, & \textit{Bosons} \end{array}$$

Choose matrix representation where

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{M} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}, \quad \mathcal{O} = \begin{pmatrix} 0 & A \\ (-1)^{2s+1}A^\dagger & 0 \end{pmatrix},$$

Note: The matrix elements here are $(2s+2) \times (2s+2)$ submatrices.

Definition:
Above Hamiltonian H_s is called a *supersymmetric relativistic arbitrary-spin Hamiltonian* if

$$M_+ A = A M_-\,, \qquad A^\dagger M_+ = M_- A^\dagger\,.$$

Note: For $s = 1/2$ this is identical to the definition of a supersymmetric Dirac Hamiltonian.

Properties:

- Consider

  $$H_s^2 = \begin{pmatrix} (-1)^{2s+1}AA^\dagger + M_+^2 & 0 \\ 0 & (-1)^{2s+1}A^\dagger A + M_-^2 \end{pmatrix}$$

  Let $m > 0$ be an arbitrary mass-like parameter and define

  $$H_+ := \frac{1}{2mc^2} AA^\dagger \geq 0\,, \qquad H_- := \frac{1}{2mc^2} A^\dagger A \geq 0\,,$$

  Define supercharges by

  $$Q := \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q^\dagger = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}$$

  and the SUSY Hamiltonian by

  $$H_{SUSY} := \frac{(-1)^{2s+1}}{2mc^2} \left(H_s^2 - \mathcal{M}^2\right) = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

  results in a $N = 2$ SUSY QM system with $W = \beta$

  $$H_{SUSY} = \{Q, Q^\dagger\}\,, \qquad \{Q, W\} = 0\,, \qquad Q^2 = 0 = (Q^\dagger)^2\,.$$

- As for the Dirac case one can show that for such supersymmetric H_s exists a Foldy–Wouthuysen transformation U which diagonalises H_s

  $$H_s^{FW} := U H_s U^\dagger = \begin{pmatrix} \sqrt{M_+^2 + (-1)^{2s+1}AA^\dagger} & 0 \\ 0 & -\sqrt{M_-^2 + (-1)^{2s+1}A^\dagger A} \end{pmatrix} = \beta|H_s|$$

  The transformation explicitly reads (without proof)

  $$U = \frac{|H_s| + \beta H_s}{\sqrt{2H_s^2 + 2\mathcal{M}|H_s|}} = \frac{1 + \beta\, \mathrm{sgn}\, H_s}{\sqrt{2 + \{\mathrm{sgn}\, H_s, \beta\}}}$$

- Due to the SUSY requirement we have $[H_\pm, M_\pm] = 0$ and we can introduce a joint set of eigenfunctions $\phi_\varepsilon^\pm$, this is a $(2s+1)$-spinor, with

$$H_\pm \phi_\varepsilon^\pm = \varepsilon \phi_\varepsilon^\pm \,, \qquad M_\pm \phi_\varepsilon^\pm = m_\varepsilon c^2 \phi_\varepsilon^\pm \,, \qquad \varepsilon \geq 0 \,.$$

Hence the spectral properties of H_s^{FM} can be expressed in terms of $\phi_\varepsilon^\pm$ and ε

$$E_\pm = \pm\sqrt{m_\varepsilon^2 c^4 + (-1)^{2s+1} 2mc^2\varepsilon} \,, \qquad \widetilde{\Psi}_\varepsilon^+ = \begin{pmatrix} \phi_\varepsilon^+ \\ 0 \end{pmatrix} , \qquad \widetilde{\Psi}_\varepsilon^- = \begin{pmatrix} 0 \\ \phi_\varepsilon^- \end{pmatrix} ,$$

The SUSY transformations explicitly read for $\varepsilon > 0$

$$\phi_\varepsilon^+ = \frac{1}{\sqrt{2mc^2\varepsilon}} A\, \phi_\varepsilon^- \,, \qquad \phi_\varepsilon^- = \frac{1}{\sqrt{2mc^2\varepsilon}} A^\dagger\, \phi_\varepsilon^+ \,.$$

The spectrum is symmetric about zerowith possible exception at m_0c^2 and/or $-m_0c^2$ in case of unbroken SUSY with $\ker A^\dagger$ and/or $\ker A$ being not empty, respectively.

Examples
We consider spin-s particles with mass $m > 0$ and charge e in external magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$.

- The Klein-Gordon Hamiltonian $s = 0$:
The non-relativistic quantum dynamics is provided by the Landau Hamiltonian

$$H_L := \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 \qquad \text{acting on} \qquad L^2(\mathbb{R}^3)$$

In 1958 Feshbach and Villars showed that the relativistic Klein-Gordon Hamiltonian is given by

$$H_0 = \begin{pmatrix} mc^2 + H_L & H_L \\ -H_L & -(mc^2 + H_L) \end{pmatrix} \qquad \text{acting on} \qquad L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$$

Obviously we may identify

$$M_\pm = H_L + mc^2 \,, \qquad A = H_L = A^\dagger \qquad \Longrightarrow \qquad [M_\pm, A] = 0$$

Hence it is a supersymmetric spin-zero Hamiltonian with

$$H_\pm = \frac{1}{2mc^2} H_L^2$$

The diagonlised FW Hamiltonian reads

$$H_0^{FW} = \begin{pmatrix} \sqrt{(mc^2 + H_L)^2 - H_L^2} & 0 \\ 0 & -\sqrt{(mc^2 + H_L)^2 - H_L^2} \end{pmatrix} = \beta mc^2 \sqrt{1 + \frac{2H_L}{mc^2}}$$

For a constant magnetic field $\vec{B} = B\vec{e}_z$ the eigenvalues of H_L are the well-know Landau levels

$$\epsilon := \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m} \,, \qquad n \in \mathbb{N}_0 \,, \qquad k_z \in \mathbb{R} \,, \qquad \omega_c := \frac{|eB|}{mc} \,.$$

Note, the eigenvalues of $H_\pm = \frac{H_L^2}{2mc^2}$ are given by $\varepsilon = \frac{\epsilon^2}{2mc^2} > 0$ and SUSY is broken. The eigenvalues of $M_\pm$ are given by $m_\varepsilon = \epsilon + mc^2 = mc^2\left(1 + \sqrt{\frac{2\varepsilon}{mc^2}}\right)$

- The Dirac Hamiltonian $s = 1/2$:
  The non-relativistic quantum dynamics is provided by the Pauli Hamiltonian with $g = 2$

$$H_P := \frac{1}{2m}\left[\vec{\sigma}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right)\right]^2 \qquad \text{acting on} \qquad L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$$

  The relativistic Dirac Hamiltonian is given by

$$H_{1/2} = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right) \\ c\vec{\sigma}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right) & -mc^2 \end{pmatrix} \qquad \text{acting on} \qquad L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$$

  We already know that it is supersymmetric with $M_\pm = mc^2$ and $A = c\vec{\sigma}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right)$.
  The partner Hamiltonians are given by

$$H_\pm = \frac{1}{2mc^2}\,A^2 = H_P$$

  The diagonlised FW Hamiltonian reads

$$H_{1/2}^{FW} = \begin{pmatrix} \sqrt{m^2c^4+2mc^2H_P} & 0 \\ 0 & -\sqrt{m^2c^4+2mc^2H_P} \end{pmatrix} = \beta mc^2\sqrt{1+\frac{2H_P}{mc^2}}$$

  For a constant magnetic field $\vec{B} = B\vec{e}_z$ the eigenvalues of H_P are shifted Landau levels

$$\varepsilon := \hbar\omega_c\left(n+\tfrac{1}{2}+s_z\right)+\tfrac{\hbar^2k_z^2}{2m}\,, \qquad n\in\mathbb{N}_0\,, \qquad k_z\in\mathbb{R}\,, \qquad s_z = \pm\tfrac{1}{2}\,.$$

  SUSY is unbroken here due to the shift!

- The vector boson Hamiltonian $s = 1$:
  The non-relativistic quantum dynamics is provided by the "vector" Hamiltonian for $g = 2$

$$H_V := \frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{mc}(\vec{S}\cdot\vec{B}) \qquad \text{acting on} \qquad L^2(\mathbb{R}^3)\otimes\mathbb{C}^3$$

  Here $\vec{S} = (S_1,S_2,S_3)^T$ are the spin-1 matrices obeying $[S_i,S_j] = \mathrm{i}\varepsilon_{ijk}S_k$,

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&1&0\\1&0&1\\0&1&0\end{pmatrix}\,, \qquad S_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&-\mathrm{i}&0\\ \mathrm{i}&0&-\mathrm{i}\\0&\mathrm{i}&0\end{pmatrix}\,, \qquad S_3 = \begin{pmatrix} 1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\,.$$

  The relativistic Hamiltonian describing a vector boson with $g = 2$ is given by

$$H_1 = \begin{pmatrix} mc^2+H_V & \frac{\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2}{2m} - \frac{\left((\vec{p}-\frac{e}{c}\vec{A})\cdot\vec{S}\right)^2}{m} \\ -\frac{\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2}{2m} + \frac{\left((\vec{p}-\frac{e}{c}\vec{A})\cdot\vec{S}\right)^2}{m} & -(mc^2+H_V) \end{pmatrix} \qquad \text{on} \qquad L^2(\mathbb{R}^3)\otimes\mathbb{C}^6$$

  With $M_\pm = mc^2 + H_V$ and $A = \frac{\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2}{2m} - \frac{\left((\vec{p}-\frac{e}{c}\vec{A})\cdot\vec{S}\right)^2}{m} = A^\dagger$ one can show that, for a **constant** magnetic field $[M_\pm, A] = 0$, leading to a supersymmetric relativistic spin-1 Hamiltonian. In addition one may show that $H_V^2 = A^2$.
  The diagonalised FW Hamiltonian then reads

$$H_1^{FW} = \begin{pmatrix} \sqrt{(mc^2+H_V)^2-H_V^2} & 0 \\ 0 & -\sqrt{(mc^2+H_V)^2-H_V^2} \end{pmatrix} = \beta mc^2\sqrt{1+\frac{2H_V}{mc^2}}$$

  The eigenvalues of $H_V = H_L - \mathrm{sgn}(eB)\,\hbar\omega_c\,S_3$ are again given by the Landau levels

$$\epsilon := \hbar\omega_c\left(n+\tfrac{1}{2}+s_z\right)+\tfrac{\hbar^2k_z^2}{2m}\,, \qquad n\in\mathbb{N}_0\,, \qquad k_z\in\mathbb{R}\,, \qquad s_z\in\{-1,0,1\}\,.$$

The partner Hamiltonians $H_\pm = \frac{1}{2mc^2} H_V^2$ have the eigenvalues $\varepsilon = \frac{\epsilon^2}{2mc^2}$.
The eigenvalues of $M_\pm$ are given by $m_\varepsilon = \epsilon + mc^2 = mc^2 \left(1 + \sqrt{\frac{2\varepsilon}{mc^2}}\right)$.

Note that $\varepsilon = 0$ when $\epsilon = 0$, which is the case for $n = 0$, $s_z = -1$ and $k_z = \pm 1/\lambda_L$.
$\lambda_L := \sqrt{\hbar/m\omega_c} = \sqrt{\hbar c/|eB|}$ is the Larmor wavelength.
Hence SUSY is unbroken, but $\Delta = 0$ as $H_+ = H_-$.

The corresponding eigenvalues of H_1 are then given by

$$E_\pm = \pm\sqrt{m^2c^4 + \hbar^2c^2k_z^2 + 2mc^2\hbar\omega_c(n + 1/2 + s_z)}$$

Note: For $k_z = 0$, $n = 0$ and $s_z = -1$, the above eigenvalue would become complex if $|B| > m^2c^3/|e|\hbar$. Such large magnetic fields would imply $\lambda_L < \lambda\!\!\!^{-}_C := \hbar/mc$. That is, the Larmor wavelength is small than the reduced Compton wavelength.

Let's confine a particle to such a small area $\Delta x \sim \lambda\!\!\!^{-}_C$.
Then uncertainty relation implies $\Delta p \sim \hbar/\Delta x = mc$. At such large energies a single particle description is no longer appropriate. In other words for such large magnetic fields a description via quantum field theory must be applied.

For details see GJ, Symmetry 12 (2020) 1590 (14pp)

Summary Section 8

Supersymmetric Dirac Hamiltonians are of the form

$$H_D = \begin{pmatrix} M_+ & A \\ A^\dagger & -M_- \end{pmatrix} \qquad \text{with} \qquad M_+ A = A M_- \,, \quad M_- A^\dagger = A^\dagger M_+ \,.$$

The $N = 2$ SUSY is explicated via ($m > 0$ is a free parameter with dimension of a mass)

$$Q = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \frac{1}{\sqrt{2mc^2}} \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix},$$

$$H_{SUSY} = \{Q, Q^\dagger\} = \frac{1}{2mc^2} \left(H_D^2 - \mathcal{M}^2\right) = \frac{1}{2mc^2} \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad W = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note

$$H_+ = \frac{AA^\dagger}{2mc^2} \,, \quad H_- = \frac{A^\dagger A}{2mc^2} \,, \quad [M_+, H_+] = 0 = [M_-, H_-]$$

Supersymmetric Dirac Hamiltonians can always be diagonalised via a FW transformation

$$H_D^{FW} = U H_D U^\dagger = \beta |H_D| = \begin{pmatrix} \sqrt{M_+^2 + 2mc^2 H_+} & 0 \\ 0 & -\sqrt{M_-^2 + 2mc^2 H_-} \end{pmatrix}.$$

The spectral properties of H_D are fully determined by those of the non-relativistic Pauli-like partner Hamiltonians $H_\pm$ and the often trivial mass operators $M_\pm$.

☺