Homework 5

Problem 1:

$$\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}$$

a) $(\vec\sigma\cdot\vec A)(\vec\sigma\cdot\vec B) = \sigma_i A_i \sigma_j B_j = \sigma_i\sigma_j A_i B_j = \delta_{ij} A_i B_j + i\epsilon_{ijk}\sigma_k A_i B_j$

$= \vec A\cdot\vec B + i\,\vec\sigma\cdot(\vec A\times\vec B)$ Note in general $[A_i, B_j] \neq 0$! $\rightsquigarrow$ ordering important

b) $(\vec\sigma\cdot\vec e_r)^2 = \underbrace{\vec e_r\cdot\vec e_r}_{=1} + i\,\vec\sigma\cdot(\underbrace{\vec e_r\times\vec e_r}_{=0}) = 1$

c) $\vec\sigma\cdot\vec p = \underbrace{(\vec\sigma\cdot\vec e_r)(\vec\sigma\cdot\vec e_r)}_{=1}(\vec\sigma\cdot\vec p) = (\vec\sigma\cdot\vec e_r)\left[\vec e_r\cdot\vec p + i\vec\sigma\cdot(\vec e_r\times\vec p)\right]$

$\vec e_r\cdot\vec p = \frac{\hbar}{i}\partial_r = -i\partial_r \quad (\hbar = 1)$

$= (\vec\sigma\cdot\vec e_r)\left(-i\partial_r + i\frac{1}{r}\vec\sigma\cdot(\underbrace{\vec r\times\vec p}_{=\vec L})\right) = -i(\vec\sigma\cdot\vec e_r)\left(\partial_r - \vec\sigma\cdot\vec L/r\right)$

d) $(\vec\sigma\cdot\vec p)(\vec\sigma\cdot\vec e_r) = \vec p\cdot\vec e_r + i\vec\sigma\cdot(\vec p\times\vec e_r) = \vec e_r\cdot\vec p - i\vec\sigma\cdot(\vec e_r\times\vec p)$

$(\vec\sigma\cdot\vec e_r)(\vec\sigma\cdot\vec p) = \vec e_r\cdot\vec p + i\vec\sigma\cdot(\vec e_r\times\vec p)$

$\Rightarrow [(\vec\sigma\cdot\vec e_r), (\vec\sigma\cdot\vec p)] = 2i\,\vec\sigma\cdot(\vec e_r\times\vec p) = \frac{2i}{r}\vec\sigma\cdot(\vec r\times\vec p) = 2i\,\vec\sigma\cdot\vec L/r$

Problem 2:

a) $\{\alpha_i, \alpha_j\} = \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}\begin{pmatrix}0 & \sigma_j\\ \sigma_j & 0\end{pmatrix} + \begin{pmatrix}0 & \sigma_j\\ \sigma_j & 0\end{pmatrix}\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix} = \begin{pmatrix}\{\sigma_i,\sigma_j\} & 0\\ 0 & \{\sigma_i,\sigma_j\}\end{pmatrix} = 2\delta_{ij}$

Note: $\sigma_i\sigma_j + \sigma_j\sigma_i = \delta_{ij} + i\epsilon_{ijk}\sigma_k + \delta_{ji} + i\underbrace{\epsilon_{jik}}_{=-1}\sigma_k = 2\delta_{ij}$

$\{\alpha_i, \beta\} = \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix} = \begin{pmatrix}0 & -\sigma_i\\ \sigma_i & 0\end{pmatrix} + \begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix} = 0$

$\beta^2 = 1$ obvious!

b) $\{\alpha_i, \alpha_j\} = \begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix}\begin{pmatrix}\sigma_j & 0\\ 0 & -\sigma_j\end{pmatrix} + \begin{pmatrix}\sigma_j & 0\\ 0 & -\sigma_j\end{pmatrix}\begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix} = \begin{pmatrix}\{\sigma_i,\sigma_j\} & 0\\ 0 & \{\sigma_i,\sigma_j\}\end{pmatrix} = 2\delta_{ij}$

$\{\alpha_i, \beta\} = \begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix} = \begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix} + \begin{pmatrix}0 & -\sigma_i\\ \sigma_i & 0\end{pmatrix} = 0$

$\beta^2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1$

c) $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$ follows from a) $\qquad \beta^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1$

$\{\alpha_i, \beta\} = \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix} = \begin{pmatrix}i\sigma_i & 0\\ 0 & -i\sigma_i\end{pmatrix} + \begin{pmatrix}-i\sigma_i & 0\\ 0 & i\sigma_i\end{pmatrix} = 0$

Problem 3 : $H_+ = \frac{AA^+}{2mc^2}$, $H_- = \frac{A^+A}{2mc^2}$

Consider:

$$AA^+ = c^2 \sigma_j (p_j - im\omega x_j)\sigma_k (p_k + im\omega x_k)$$

$$= c^2 \sigma_j \sigma_k \left(p_j p_k + im\omega (p_j x_k - x_j p_k) + m^2\omega^2 x_j x_k \right)$$

$$= c^2 \left(\vec{p}^2 + m^2\omega^2 \vec{r}^2 \right) + ic^2 m\omega\, \delta_{jk} \underbrace{(p_j x_k - x_j p_k)}_{= \frac{\hbar}{i}\delta_{jk}} + ic^2 m\omega\, i\varepsilon_{jkl}\sigma_l \underbrace{(p_j x_k - x_j p_k)}_{\substack{= -2x_j p_k \\ \text{as } j\neq k}}$$

$$= c^2 (\vec{p}^2 + m^2\omega^2\vec{r}^2) + mc^2\hbar\omega \underbrace{\delta_{jk}\delta_{jk}}_{=3} + 2mc^2\omega \underbrace{\varepsilon_{jkl}\sigma_l x_j p_k}_{\vec{\sigma}\cdot\vec{L}}$$

$$= c^2 (\vec{p}^2 + m^2\omega^2\vec{r}^2) + 3mc^2\hbar\omega + 2mc^2\omega\,\vec{\sigma}\cdot\vec{L}$$

$$= 2mc^2 \left(\frac{\vec{p}^2}{2m} + \frac{m}{2}\omega^2\vec{r}^2 \right) + 2mc^2 \left(\frac{3}{2}\hbar\omega + 2\frac{\omega}{\hbar}\vec{S}\cdot\vec{L} \right) \qquad \vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

with $K := \frac{2}{\hbar^2}\vec{L}\cdot\vec{S} + 1 \;\Leftrightarrow\; \vec{L}\cdot\vec{S} = \frac{K-1}{2}\hbar^2$

$$= 2mc^2 \left(\frac{\vec{p}^2}{2m} + \frac{m}{2}\omega^2\vec{r}^2 + \hbar\omega (K + \tfrac{1}{2}) \right)$$

A^+A follows from AA^+ with $\omega \to -\omega$

$$\Rightarrow \quad \underline{\underline{H_\pm = \left[\frac{\vec{p}^2}{2m} + \frac{m}{2}\omega^2\vec{r}^2 \pm \hbar\omega (K + \tfrac{1}{2}) \right]}}$$