4. Homework

Problem 1: From lecture known

SUSY potential: $\Phi : \mathbb{R} \to \mathbb{R}$

Drift potentials: $U_\pm(x) := \mp \int dx\, \Phi(x)$

Fokker-Planck Hamiltonian:

$$H_\pm^{FP} := -\frac{1}{2}\partial_x^2 + V_\pm^{FP}(x), \quad V_\pm^{FP}(x) = \frac{1}{2}\Phi^2(x) \pm \frac{1}{2}\Phi'(x)$$

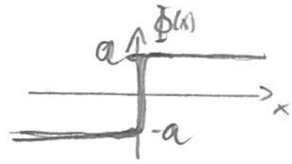
with $\quad H_\pm^{FP} |\phi_n^\pm\rangle = \lambda_n |\phi_n^\pm\rangle \quad n = 1,2,3 \quad \lambda_n > 0$

and in case of unbroken SUSY $\Delta = +1$ (convention)

$$H_-^{FP} |\phi_0^-\rangle = 0 \qquad (\text{i.e. } \lambda_0 = 0)$$

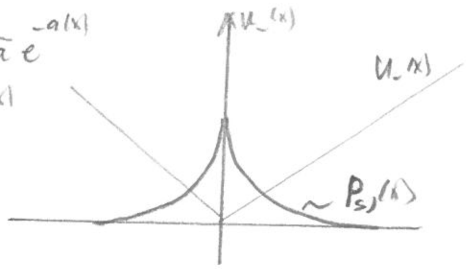
with $\quad P_{st}(x) = |\phi_0^-(x)|^2$ stationary distribution

Case 1: $\Phi_1(x) = a\,\mathrm{sgn}(x)$ SUSY unbroken with $\Delta = +1$ as $a > 0$



$$\phi_0^-(x) = N \exp\{-\int dx\, \Phi_1(x)\} = N e^{-a|x|} = \sqrt{a}\, e^{-a|x|}$$

$$\curvearrowright P_{st}(x) = |\phi_0^-(x)|^2 = N^2 \exp\{-2a|x|\} = a\, e^{-2a|x|}$$

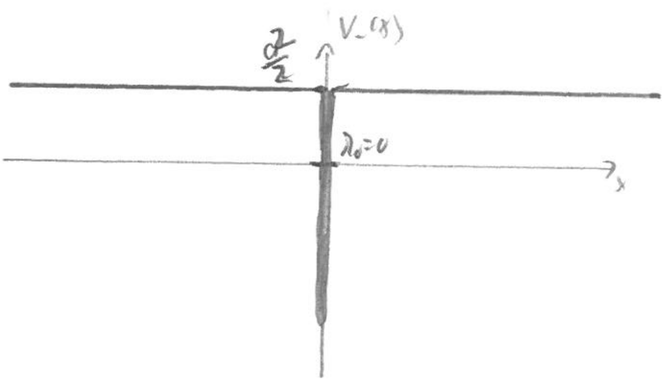


$$U_-(x) = \int dx\, \Phi_1(x) = a|x| = -U_+(x)$$

$$V_\pm^{FP}(x) = \frac{a^2}{2} \pm a\,\delta(x) \quad (\text{only 1 bound state for } V_-)$$

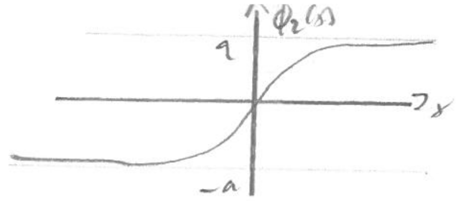
linear confining pot.!

$$\curvearrowright \lambda_1 = \min \operatorname{spec}(H_+^{FP}) = \frac{a^2}{2}$$



Case 2: $\Phi_2(x) = a \tanh x$, $a>0$

SUSY unbroken with $\Delta = +1$ as $a>0$

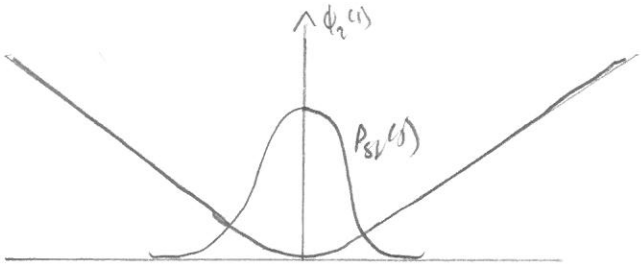


$$\phi_0^-(x) = \mathcal{N} \exp\{-\int dx\, \Phi(x)\} = \frac{\mathcal{N}}{\cosh^a x}$$

$$P_{st}(x) = \frac{\mathcal{N}^2}{\cosh^{2a} x}$$

$$U_-(x) = \int dx\, \Phi_2(x) = a \ln \cosh x$$

$$V_\mp^{FP}(x) = \frac{a^2}{2} - \frac{a(a\mp 1)}{\cosh^2 x}$$



lin. confining potential with smooth edge !

shape-invariant ↗ Section 3.3

$$\rightsquigarrow \quad \lambda_n = \frac{a^2}{2} - \frac{1}{2}(a-n)^2 \qquad n = 0, 1, 2, \dots < a$$

$\lambda_0 = 0$ obvious as SUSY is unbroken

$a>1$: $\quad \lambda_1 = \frac{a^2}{2} - \frac{1}{2}(a-1)^2 = a + \frac{1}{2}$

$a<1$: $\quad \lambda_1 = \frac{a^2}{2}$

In general:

λ_1 defines the largest time scale in which a general initial probability distribution decays for U_+ or assumes the stationary distribution $P_{st}(x)$

Remember: $e^{-\lambda_1 t} \rightsquigarrow \boxed{\tau = 1/\lambda_1}$ decay time

Problem 2:

Assume $H_0^{\pm} := -\frac{1}{2}\partial_x^2 + \frac{1}{2}\Phi^2(x) \pm \frac{1}{2}\Phi'(x)$ unperturbed Hamiltonian

where spectral properties are known

$$H_0^{\pm}|\phi_n^{\pm}\rangle = \lambda_n|\phi_n^{\pm}\rangle \qquad n=1,2,3,\dots \qquad \lambda_n > 0$$

and

$$H_0^-|\phi_0^-\rangle = 0 \qquad \text{for unbroken SUSY}$$

For example: $\Phi(x)$ is one of the shape-inv. SUSY potentials

Now with perturbation: $W(x) := \Phi(x) + \underbrace{f(x)}_{\text{"perturbation"}}$

$$\rightsquigarrow \quad V_\pm^{FP}(x) = \frac{1}{2}W^2(x) \pm \frac{1}{2}W'(x) = \frac{1}{2}(\Phi^2 + 2\Phi f + f^2) \pm \frac{1}{2}(\Phi' + f')$$

$$= \frac{1}{2}\Phi^2(x) \pm \frac{1}{2}\Phi'(x) + \frac{1}{2}\left(f^2(x) + 2\Phi(x)f(x) \pm f'(x)\right)$$

- Assumption: $\boxed{f^2(x) + 2f(x)\Phi(x) + f'(x) = b = \text{const}}$ (1)

$$\rightsquigarrow \quad V_+^{FP}(x) = \frac{1}{2}\Phi^2(x) + \frac{1}{2}\Phi'(x) + \frac{b}{2} \qquad \rightsquigarrow \quad H_+^{FP} = H_0^+ + \frac{b}{2}$$

$\rightsquigarrow$ spectral properties are known as only EV are shifted to $\lambda_n + \frac{b}{2}$!

$$V_-^{FP}(x) = \frac{1}{2}\Phi^2(x) - \frac{1}{2}\Phi'(x) + \frac{b}{2} - f'(x)$$

Hence: $H_+^{FP}|\phi_n^+\rangle = (\lambda_n + \frac{b}{2})|\phi_n^+\rangle$ With help of SUSY Trafo also spectral prop. of H_-^{FP} are known!

Ansatz: $f(x) = \frac{V'(x)}{V(x)} \quad \rightsquigarrow \quad f'(x) = \frac{V''(x)}{V(x)} - \left(\frac{V'(x)}{V(x)}\right)^2$ in (1)

$$\cancel{\frac{V'^2}{V^2}} + 2\frac{V'}{V}\Phi + \frac{V''}{V} - \cancel{\frac{V'^2}{V^2}} = b$$

$$\rightsquigarrow \quad \boxed{V'' + 2\Phi V' = bV} \quad (2)$$

Ansatz: $u(x) = v(x) \exp\left\{ \int dx\, \Phi(x) \right\}$

$\rightsquigarrow \quad v(x) = u(x) \exp\left\{ -\int dx\, \Phi(x) \right\}$

$\rightsquigarrow \quad v' = \frac{u'}{u} v - \Phi v = v\left(\frac{u'}{u} - \Phi\right)$

$\rightsquigarrow \quad v'' = v\left(\frac{u'}{u} - \Phi\right)^2 + v\left(\frac{u''}{u} - \frac{u'^2}{u^2} - \Phi'\right)$ in 2

$$\cancel{v}\left(\frac{u'}{u} - \Phi\right)^2 + \cancel{v}\left(\frac{u''}{u} - \frac{u'^2}{u^2} - \Phi'\right) + 2\Phi \cancel{v}\left(\frac{u'}{u} - \Phi\right) = b\cancel{v} \quad | v \neq 0$$

$$\cancel{\frac{u'^2}{u^2}} - \cancel{2\Phi \frac{u'}{u}} + \Phi^2 + \frac{u''}{u} - \cancel{\frac{u'^2}{u^2}} - \Phi' + \cancel{2\Phi \frac{u'}{u}} - 2\Phi^2 = b$$

$$\frac{u''}{u} - \Phi^2 - \Phi' = b$$

or
$$\boxed{\left(-\frac{1}{2} d_x^2 + \frac{1}{2}\Phi^2 + \frac{1}{2}\Phi'\right) u = -\frac{b}{2} u}$$

$H_0^+ u(x) = \varepsilon\, u(x)$ Schrödinger-like equation

with $\varepsilon = -\frac{b}{2}$ again $u \in L^2$ NOT required!

In fact $u(x) \neq 0 \ \forall x \in \mathbb{R}$ to avoid singularities

$\rightsquigarrow \quad \varepsilon < \min \operatorname{spec}(H_0^+) = \lambda_1 \quad \Rightarrow \quad \boxed{\varepsilon < \lambda_1}$ or $\boxed{b > -2\lambda_1}$

Drift potential:

$$U_\pm(x) = \mp \int dx\, W(x) = \mp \int dx \left(\Phi(x) + f(x)\right) = \mp \int dx \left(\Phi(x) + \frac{v'(x)}{v(x)}\right)$$

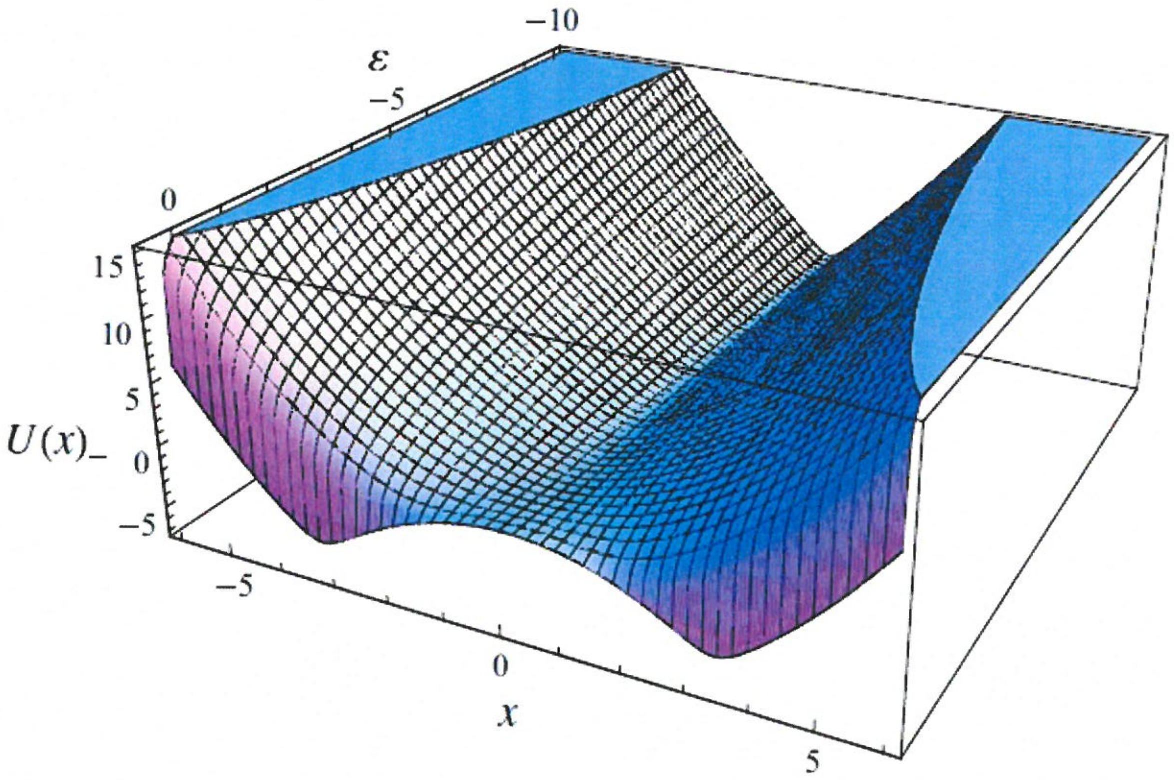
$$= \mp \int dx\, \Phi(x) \mp \ln v(x) = \mp \int dx\, \Phi(x) \mp \ln u(x) \pm \int dx\, \Phi(x)$$

$$= \mp \ln u(x)$$

New conditionally exactly solvable drift potentials!

Example $\Phi(x) = x$

$$\Rightarrow U_\pm(x) = \pm\frac{x^2}{2} \mp \ln\left[{}_1F_1\left(\frac{1-2\epsilon}{4}, \frac{1}{2}, x^2\right) + \beta x \; {}_1F_1\left(\frac{3-2\epsilon}{4}, \frac{3}{2}, x^2\right)\right]$$



$\beta = 0$