# 3. Homework



Problem 1: According to Section 4.2 for $\varepsilon = -\frac{1}{2} - 2k$, $\beta = 0$ the solution reads

$$u(x) = e^{x^2/2} H_{2k}(ix) \qquad \text{here } k=1 \rightsquigarrow H_2(z) = 4z^2 - 2 \quad \text{2. Hermite polynomial}$$

Hence $\quad u(x) = (-4x^2-2)\, e^{x^2/2}$

$$u'(x) = x\, u(x) + e^{-x^2/2}(-8x) = x\, u(x) + \frac{8x}{4x^2+2} u(x)$$

$$\rightsquigarrow \frac{u'(x)}{u(x)} = x + \frac{4x}{2x^2+1}$$

With $\quad V_-(x) = \left(\frac{u'(x)}{u(x)}\right)^2 - \frac{x^2}{2} + 2\varepsilon$

$$= \left(x + \frac{4x}{2x^2+1}\right)^2 - \frac{x^2}{2} - 5 = \underline{\underline{\frac{x^2}{2} + \frac{8x^2}{2x^2+1} + \left(\frac{4x}{2x^2+1}\right)^2 - 5}}$$

$$\mathrm{Spec}(H_-) = \left\{-\frac{5}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\right\}$$

Compare with general formula for arbitrary $k \in \mathbb{N}$

$$V_-(x) = \frac{x^2}{2} + 2ix \frac{H_{2k+1}(ix)}{H_{2k}(ix)} - \left(\frac{H_{2k+1}(ix)}{H_{2k}(ix)}\right)^2 - 4k - 1$$

Note $\quad H_3(ix) = -i\,8x^3 - i\,12x$

$$2ix H_3(ix) = 16x^4 + 24x^2 = 8x^2(2x^2+3)$$

$$\rightsquigarrow \frac{2ix H_3(ix)}{H_2(ix)} = 8x^2 \frac{2x^2+3}{-4x^2-2} = -4x^2 \frac{2x^2+3}{2x^2+1} \, .$$

$$\left(\frac{H_3(ix)}{H_2(ix)}\right)^2 = \left(\frac{8ix^3 + 12ix}{4x^2+2}\right)^2 = -4x^2 \left(\frac{2x^2+3}{2x^2+1}\right)^2 \, .$$

$$V_-(x) = \frac{x^2}{2} - 4x^2 \frac{2x^2+3}{2x^2+1} + 4x^2 \frac{(2x^2+3)^2}{(2x^2+1)^2} - 5$$

$$= \frac{x^2}{2} + \frac{4x^2}{(2x^2+1)^2}(2x^2+3)(2x^2+3 - 2x^2 - 1) = \frac{x^2}{2} + \frac{4x^2}{(2x^2+1)^2}(2x^2+3)\,2 - 5$$

$$= \frac{x^2}{2} + \frac{8x^2}{(2x^2+1)^2}(2x^2+1+2) - 5 = \frac{x^2}{2} + \frac{8x^2}{2x^2+1} + \frac{16x^2}{(2x^2+1)^2} \quad \#$$

**Problem 2:** Schrödinger like eq for $V_+(x)=0$ and $\varepsilon = -\frac{\hbar^2 k^2}{2m} < 0 = E_0$ ②

$$-\frac{\hbar^2}{2m}\partial_x^2 u(x) = -\frac{\hbar^2 k^2}{2m} u(x) \implies u''(x) = k^2 u(x)$$

General solution $\quad u(x) = \alpha e^{kx} + \beta e^{-kx} \quad$ or w/o loss of generality $\alpha = 1$

$$\underline{\underline{u(x) = e^{kx} + \beta e^{-kx}}}$$

Check: $u(x)=0 \quad \leadsto \quad e^{kx} = -\beta e^{-kx} \quad \leadsto \quad \beta = -e^{2kx} \leq 0$

Hence $\quad \underline{\underline{\beta > 0}} \quad \implies \quad u(x) > 0$

Potential:

$$V_-(x) = \frac{\hbar^2}{m}\left(\frac{u'(x)}{u(x)}\right)^2 - V_+(x) + 2\varepsilon \qquad , \quad u'(x) = k\left(e^{kx} - \beta e^{-kx}\right)$$

$$= \frac{\hbar^2 k^2}{m}\left(\frac{e^{kx} - \beta e^{-kx}}{e^{kx} + \beta e^{-kx}}\right)^2 - \frac{\hbar^2 k^2}{m}$$

$$= \underline{\underline{\frac{\hbar^2 k^2}{m}\left[\left(\frac{e^{kx} - \beta e^{-kx}}{e^{kx} + \beta e^{-kx}}\right)^2 - 1\right]}} \quad \#$$

has 1 bound state $\varepsilon = -\frac{\hbar^2 k^2}{2m}$

with $\psi_\varepsilon^-(x) = \frac{c}{e^{kx} + \beta e^{-kx}}$ #

Special case $\beta = 1$:

$$\frac{e^{kx} - e^{-kx}}{e^{kx} + e^{-kx}} = \tanh kx$$

$$\leadsto \quad V_-(x) = \frac{\hbar^2 k^2}{m}\left(\tanh^2 kx - 1\right) = \frac{\hbar^2 k^2}{2m}\,\frac{2}{\cosh^2 kx}$$

Problem 3:

$$E[\phi] = \int dx \left[\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\phi'^2 + U(\phi)\right]$$

$$\frac{dE}{dt} = \int dx \left[\tfrac{1}{2}\partial_t \dot{\phi}^2 + \tfrac{1}{2}\partial_t \phi'^2 + \partial_t U(\phi)\right]$$

$$= \int dx \left[\tfrac{1}{2} 2\dot{\phi}\ddot{\phi} + \tfrac{1}{2} 2\phi'\dot{\phi}' + U'(\phi)\dot{\phi}\right]$$

$$= \int dx \left[\dot{\phi}\ddot{\phi} + \phi'\dot{\phi}' + U'(\phi)\dot{\phi}\right]$$

$$\int dx\, \phi'\dot{\phi}' = \underbrace{\phi'\dot{\phi}\Big|_{-\infty}^{+\infty}}_{=0} - \int dx\, \phi''\dot{\phi}$$

$$= \int dx\, \underbrace{\left[\ddot{\phi} - \phi'' + U'(\phi)\right]}_{=0}\dot{\phi} = 0$$

## Problem 4:

$$W(x) = \operatorname{sgn} x \quad \rightsquigarrow \quad V_-(x) = W^2(x) - W'(x) = 1 - 2\delta(x)$$

Hint:
$$\int_{-\infty}^{z} dx\, \partial_x \operatorname{sgn} x = \operatorname{sgn} x \Big|_{-\infty}^{z} = \operatorname{sgn} z + 1 = \begin{cases} 0 & z<0 \\ 2 & z>0 \end{cases}$$

$$\int_{-\infty}^{z} dx\, 2\delta(x) = \begin{cases} 0 & z<0 \\ 2 & z>0 \end{cases} \qquad \#$$

$$\psi_0(x) = N \exp\left\{ - \int dx\, W(x) \right\} = N e^{-|x|}$$

Proof:
$$\int_0^z dx\, \operatorname{sgn} x = \begin{cases} z & z>0 \\ -(-z) & z<0 \end{cases} = |z|$$

$$\phi_{st}(x) = \int dx\, \psi_0(x) = \operatorname{sgn} x \left(1 - e^{-|x|}\right) \quad \text{with } N=1 \qquad \phi_\pm = \pm 1$$

Proof:
$$\phi_{st}'(x) = \underbrace{2\delta(x)\left(1-e^{-|x|}\right)}_{=0} + \operatorname{sgn} x \left(e^{-|x|}\right) \operatorname{sgn} x = e^{-|x|}$$

Hence:
$$U(\phi_{st}) = \frac{1}{2}\left(\phi_{st}'(x)\right)^2 = \frac{1}{2} e^{-2|x|}$$

Consider $|\phi_{st}(x)| = 1 - e^{-|x|} \rightsquigarrow e^{-|x|} = 1 - |\phi_{st}(x)|$

$$\Longrightarrow \quad U(\phi_{st}) = \frac{1}{2}\left(1 - |\phi_{st}(x)|\right)^2$$

analytically continue beyond $\phi_\pm = \pm 1$

$$U(\phi) = \frac{1}{2}\left(1 - |\phi|\right)^2$$